ON BORSUK'S PASTE JOB AND RELATED TOPICS

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INTRODUCTION

The idea of Borsuk's Paste Job was first introduced in 1935 by Borsuk [1] who called it a singular retract. Other authors [3], [6] have referred to Borsuk's Paste Job as an adjunction space. At the present time, relatively little is known about the structure of these spaces; however, Chapters II and III of this thesis contain some new information. Chapter IV and V deal with the study and application of an idea which arises in Chapter III.

Chapter I is an introductory chapter, part of which contains preliminary definitions and propositions which are needed in the other chapters. The remainder of the chapter is devoted to some propositions which are developed in order to deal with certain examples.

In Chapter II it is proved that if two Borsuk's Paste Jobs have homotopic defining maps then the two Borsuk's Paste Jobs have isomorphic cohomology groups. By using a generalization of the definition of homotopic maps, a generalization of this result is also obtained.

If the construction of Borsuk's Paste Job is done on a (topological) semigroup, then does the resulting Borsuk's Paste Job admit a natural semigroup structure is the question which is asked in Chapter III. The chapter studies a particular aspect of this question, namely, the determination of

necessary or sufficient conditions for an affirmative answer to the above question for all possible defining homomorphisms on a fixed subsemigroup. Section 1 contains some necessary conditions and an indication of why attention has been restricted to homomorphisms and subsemigroups; Section 2 gives several sufficient conditions of the above type; Section 3 contains some examples and counter-examples; Section 4 gives necessary and sufficient conditions for an affirmative answer under the hypothesis that the subsemigroup consists of left zeros; Section 5 extends the results of Section 4 to cover the case of minimal ideals; and Section 6 is an investigation of the connection of the question with congruences.

The idea of a homomorphic retract, which is introduced in Chapter III, is studied further in Chapter IV. Sufficient conditions for various types of subsemigroups to be homomorphic retracts are given. In particular, two sets of necessary and sufficient conditions for the minimal ideal to be a homomorphic retract are presented.

In Chapter V, under the hypothesis that T is a homomorphic retract, the retraction properties of minimal T-ideals [17], [18], [19] are studied. Sufficient conditions are given to insure that the minimal T-ideals are retracts.

CHAPTER I

PRELIMINARY PROPOSITIONS AND DEFINITIONS

0: Quotient Spaces.

A relation on a set X is a subset of $X \times X$. If R is a relation on X and if A is a subset of X, then the following notation is used: $AR = \{y \mid (x,y) \in R \text{ for some } x \in A\}$, $R^{(-1)} = \{(y,x) \mid (x,y) \in R\}$, $RA = AR^{(-1)}$, $R \circ R = \{(x,z) \mid (x,y) \text{ and } (y,z) \in R \text{ for some } y \in X\}$. Let R be a relation on X, then R is reflexive in case $\Delta X = \{(x,x) \mid x \in X\}$ is a subset of R, R is symmetric if and only if $R^{(-1)} = R$, and R is transitive whenever $R \circ R$ is a subset of R. An equivalence relation is a reflexive, symmetric and transitive relation. If R is an equivalence relation on X, then $\{xR \mid x \in X\}$ is called the factor set of X by R and is denoted by X/R. The function φ from X to X/R defined by $\varphi(x) = xR$ is called the natural projection of X onto X/R.

If X is a topological space and R is an equivalence relation on X, then X/R is topologized by the smallest topology for which the natural projection is continuous. More explicitly, a subset U of X/R is open if and only if $\varphi^{-1}(U)$ is open. This topology on X/R is called the quotient topology [7] and it is assumed, without further mention, that all factor sets of topological spaces are endowed with the

quotient topology.

1.0.1: Lemma. If R is an equivalence relation on a topological space X and if f is a function whose domain is X/R, then f is continuous if and only if $f\varphi$ is continuous.

Proof: See [7, p. 96].

If X is a Hausdorff space and R is an equivalence relation which is defined on X, then X/R need not be Hausdorff [5, p. 132]. In the next lemma conditions are given which insure that X/R is Hausdorff.

1.0.2: Lemma. If X is a compact Hausdorff space and R is a closed equivalence relation on X, then X/R is Hausdorff. This is a direct consequence of [12, Theorem 9] and [5, p. 13].

1: Definition of Borsuk's Paste Job.

The type of construction described here was first given by Karol Borsuk [1] and was called a singular retract.

Throughout this section, X and Y are compact Hausdorff spaces, A is a closed subspace of X, and f is a continuous function from A onto Y.

1.1.1: Definition. (a) If $f \times f$ is the function from $A \times A$ to $Y \times Y$ defined by $(f \times f)(a,b) = (f(a),f(b))$, then $R(f,X) = [(f \times f)^{-1}(\Delta Y)] \cup \Delta X$. It is easy to see that $R(f,X)$ is a closed equivalence relation on X and that $(x,y) \in R(f,X)$ if and only if $f(x) = f(y)$ or $x = y$.

(b) Let $Z(f,X) = X/R(f,X)$. Then $Z(f,X)$ is called Borsuk's Paste Job of Y to X by f. It

follows from Lemma 1.0.2 and 1.1.1 (a) that $Z(f,X)$ is a compact Hausdorff space.

The following lemma is easily proved using 1.1.1.

<u>1.1.2: Lemma.</u> If φ is the natural projection of X onto $Z(f,X)$, then there exists a function k from Y into $Z(f,X)$ such that $kf = \varphi i$, where i is the injection of A into X, i.e., the following diagram is analytic.

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Z(f,x) \\ A \arrow[u, "i"] \arrow[r, "f"] & Y \arrow[u, "k"'] \end{tikzcd}$$

Moreover, k is a homeomorphism of Y onto $\varphi(A)$ and φ maps $(X - A)$ homeomorphically onto $(Z(f,X) - k(Y))$.

This lemma says that $Z(f,X) = \varphi(X - A) \cup k(Y)$ and that $\varphi(X - A)$ is topologically the same as $(X - A)$ and $k(Y)$ is topologically the same as Y. This does not imply that the structure of $Z(f,X)$ is completely determined by Lemma 1.1.2, for let X be the unit disk E^2 and let $A = Y$ be the boundary of E^2, i.e., S^1. If $f_1(z) = z$ for all $z \in S^1$ and $f_2(z) = z^2$ for all $z \in S^1$, then $Z(f_1,E^2)$ is topologically E^2 and $Z(f_2,E^2)$ is the projective plane.

<u>2: Preliminary Propositions on the Cohomological Structure of Borsuk's Paste Job.</u>

All of the cohomology groups in this thesis are the Alexander-Kolmogoroff Cohomology groups [11], [14].

<u>1.2.1: Lemma.</u> Let X and Y be topological spaces, let A

and B be closed subsets of X and Y, respectively, let f be a continuous function from (X,A) to (Y,B), let the continuous function g from X to Y be defined by g(x) = f(x), and let the continuous function h from A to B be defined by h(x) = f(x). If any two of the three induced homomorphisms f*, g*, and h* are isomorphisms, then the third is also an isomorphism [4].

Proof: Suppose f* and g* are isomorphisms. In the following analytic diagram, j, j', i, and i' are inclusion mappings, δ and δ' are co-boundary operators, and the rows are exact [14].

$$\begin{array}{ccccccccc}
H^p(Y,B) & \xrightarrow{j^*} & H^p(Y) & \xrightarrow{i^*} & H^p(B) & \xrightarrow{\delta} & H^{p-1}(Y,B) & \xrightarrow{j^*} & H^{p-1}(Y) \\
\downarrow f^* & & \downarrow g^* & & \downarrow h^* & & \downarrow f^* & & \downarrow g^* \\
H^p(X,A) & \xrightarrow{j^*} & H^p(X) & \xrightarrow{i'^*} & H^p(A) & \xrightarrow{\delta'} & H^{p-1}(X,A) & \xrightarrow{j'^*} & H^{p-1}(X)
\end{array}$$

Since f* and g* are isomorphisms, the Five Lemma [9] says that h* is also an isomorphism.

The other two cases are proven in a similar fashion, provided that $H^p(Z,K)$ is interpreted as $\{0\}$ whenever $p < 0$.

In the remainder of this chapter, X and Y are compact Hausdorff spaces, A is a closed subset of X, f is a continuous function from A onto Y, φ is the natural projection of X onto Z(f,X), and k is the homeomorphism given by Lemma 1.1.2. With this notation, the following diagram is analytic.

$$\begin{tikzcd} X \arrow[r, "\varphi"] & Z(f,X) \\ A \arrow[u, phantom, "\cup" description] \arrow[r, "f"] & Y \arrow[u, "k"'] \end{tikzcd}$$

<u>1.2.2: Lemma.</u> The equation $\varphi(x) = \varphi(x)$ defines a continuous function φ' from (X,A) onto $(Z(f,X), k(Y))$. The induced homomorphism φ'^* is an isomorphism.

The lemma follows immediately from Lemma 1.1.2 and the Map Excision Theorem [13].

This lemma says that for the proper choice of homomorphisms there is an exact sequence: $H^0(X,A) \longrightarrow H^0(Z(f,X)) \longrightarrow H^0(Y) \longrightarrow H^1(X,A) \longrightarrow \ldots \longrightarrow H^{p-1}(Y) \longrightarrow H^p(X,A) \longrightarrow H^p(Z(f,X)) \longrightarrow H^p(Y) \longrightarrow H^{p+1}(X,A) \longrightarrow \ldots$.

<u>1.2.3: Lemma.</u> The induced homomorphism φ^* is an isomorphism if and only if f^* is an isomorphism.

Proof: Let φ' be as in Lemma 1.2.2 and φ'' be the continuous function from A to k(Y) which is defined by $\varphi''(x) = \varphi(x)$ for all $x \in A$. By Lemma 1.1.2 it follows that $\varphi''^* = (kf)^* = f^*k^*$. Thus φ''^* is an isomorphism if and only if f^* is an isomorphism. According to Lemma 1.2.1, φ''^* is an isomorphism if and only if φ^* is an isomorphism, and so φ^* is an isomorphism if and only if f^* is an isomorphism.

<u>1.2.4: Lemma.</u> $k(Y)$ is a retract of $Z(f,X)$ if and only if f is extendable to X.

Proof: See [6, p. 10].

<u>1.2.5: Corollary.</u> If f is extendable to X, then $H^p(Z)$ is

isomorphic to $H^p(X,A) \times H^p(Y)$.

Proposition 1.2.9 provides a method for applying the Absolute Mayer-Vietorsis Sequence [14] to Borsuk's Paste Job. First, a few lemmas which are needed in Proposition 1.2.9 are proved.

<u>1.2.6: Lemma.</u> Let B be a closed subset of X, let $A' = B \cap A$ and let f' be the restriction of f to A'. Then $Z(f',B)$ can be topologically embedded in $Z(f,X)$.

Proof: Define h from $Z(f',B)$ into $Z(f,X)$ by $h(z) = \phi\phi'^{-1}(z)$ for all $z \in Z(f',B)$ where ϕ' is the natural projection of B onto $Z(f',B)$. It is enough to show that h is a homeomorphism.

To see that h is well defined, let $x,y \in B$ such that $\phi'(x) = \phi'(y)$. By 1.1.1, $x = y$ or $f'(x) = f'(y)$ so that $x = y$ or $f(x) = f(y)$ and $\phi(x) = \phi(y)$. Lemma 1.0.1 implies h is continuous since $h\phi'$ is ϕ restricted to B. Thus it remains only to show that h is one-to-one. Let $x,y \in B$ such that $h\phi'(x) = h\phi'(y)$. Then $\phi(x) = h\phi'(x) = h\phi'(y) = \phi(y)$ so that $x = y$ or $f(x) = f(y)$. It follows that $\phi'(x) = \phi'(y)$.

<u>1.2.7: Lemma.</u> Let B_1 and B_2 be closed subsets of X, let $A_i = B_i \cap A$, let f_i be the restriction of f to A_i, and let h_i be defined as in Lemma 1.2.6, where $i = 1$ or 2. If $B_1 \cup B_2 = X$, then $h_1(Z(f_1,B_1) \cup h_2(Z(f_2,B_2)) = Z(f,X)$.

Proof: If $z \in Z(f,X)$, then there exists $x \in X$ such that $\phi(x) = z$. But $x \in B_1$ or $x \in E_2$ so that $z = h_i\phi_i(x)$ where ϕ_i is the natural projection of B_i onto $Z(f_i,B_i)$ and $i = 1$ or 2.

<u>1.2.8: Lemma.</u> Let B_1 and B_2 be closed subsets of S, let $B_3 = B_1 \cap B_2$, let $A_i = B_i \cap A$, let f_i be the restriction of f to A_i, and let h_i be as defined in Lemma 1.2.6, where $i = 1, 2,$ or 3. If $f(A_3) = f(A_1) \cap f(A_2)$, then $h_3(Z(f_3,B_3)) = h_1(Z(f_1,B_1)) \cap h_2(Z(f_2,B_2))$.

Proof: If $h_3(Z) \varepsilon h_3(Z(f_3,B_3))$, then there exists a point x in B_3 such that $\varphi_3(x) = z$. Thus $h_3(z) \varepsilon h_1(Z(f_1,B_1)) \cap h_2(Z(f_2,B_2))$ since $h_3(z) = \varphi(x) = h_1\varphi_1(x) = h_2\varphi_2(x)$.

If $z \varepsilon h_1(Z(f_1,B_1)) \cap h_2(Z(f_2,B_2))$, then there exist $x_1 \varepsilon B_1$ and $x_2 \varepsilon B_2$ such that $\varphi(x_1) = z = \varphi(x_2)$. Thus $x_1 = x_2$, or $f(x_1) = f(x_2)$. If $x_1 = x_2$, then $x_1 \varepsilon B_3$ and $h_3\varphi_3(x_1) = \varphi(x_1) = z$. If $f(x_1) = f(x_2)$, then there exists $x_3 \varepsilon A_3$ such that $f(x_3) = f(x_2) = f(x_1)$. Hence $h_3\varphi_3(x_3) = \varphi(x_3) = \varphi(x_1) = z$.

<u>1.2.9: Proposition.</u> Let B_1 and B_2 be closed subsets of X, let $B_3 = B_1 \cap B_2$, $A_i = B_i \cap A$, and let f_i be the restriction of f to A_i. If $B_1 \cup B_2 = X$ and $f(A_3) = f(A_1) \cap f(A_2)$, then the following sequence with the obvious choice of homomorphisms is exact.

$$H^0(Z(f,X)) \longrightarrow \begin{matrix} H^0(Z(f_1,B_1)) \\ \times \\ H^0(Z(f_2,B_2)) \end{matrix} \longrightarrow H^0(Z(f_3,B_3)) \longrightarrow$$

$$H^1(Z(f,X)) \longrightarrow \quad \cdots \quad \longrightarrow H^{p-1}(Z(f_3,B_3)) \to$$

$$H^p(Z(f,X)) \longrightarrow \begin{matrix} H^p(Z(f_1,B_1)) \\ \times \\ H^p(Z(f_2,B_2)) \end{matrix} \longrightarrow H^p(Z(f_3,B_3)) \longrightarrow$$

$$H^{p+1}(Z(f,X)) \longrightarrow \quad \ldots$$

This proposition is an immediate consequence of Lemmas 1.2.7 and 1.2.8 and the statement of the Absolute Meyer-Vietorsis Sequence.

If E^2 is the unit disk, if S^1 is the boundary of E^2, and if f is a continuous function from S^1 onto S^1, then $H^p(Z(f,E^2))$ is easily computed for each positive integer p. This computation will follow from the next two lemmas.

Notation: If A and B are groups and if g is a homomorphism from A into B, then K[g] denotes the kernel of g and I[g] denotes the image of g. The continuous function φ' from (X,A) to (Z(f,X),k(y)) is defined by $\varphi'(x) = \varphi(x)$. This notation will be used in the remainder of this section.

<u>1.2.10: Lemma.</u> (i) If $H^p(Y) = 0$ and $p > 0$, then $H^p(Z(f,X))$ is isomorphic to $H^p(X,A)/\delta f^* H^{p-1}(Y)$, where δ is the co-boundary operator from $H^{p-1}(A)$ to $H^p(X,A)$.

(ii) If $H^0(Y) = 0$, then $H^0(Z(f,X))$ is isomorphic to $H^0(X,A)$.

Proof: In the following analytic diagram i and i' are injections, δ and δ' are co-boundary operators.

$$\begin{array}{ccccccc}
H^{p-1}(k(Y)) & \xrightarrow{\delta'} & H^p(Z(f,X),k(Y)) & \xrightarrow{i'^*} & H^p(Z(f,X)) & \longrightarrow & 0 \\
\downarrow k^* & & & & & & \\
H^{p-1}(Y) & & \downarrow \varphi'^* & & \downarrow \varphi^* & & \\
\downarrow f^* & & & & & & \\
H^{p-1}(A) & \xrightarrow{\delta} & H^p(X,A) & \xrightarrow{i^*} & H^p(X) & &
\end{array}$$

Since the fact that the rows of the diagram are exact implies i'^* is onto, $H^p(Z(f,X))$ is isomorphic to $H^p(Z(f,X),$ $k(Y))/K[i^*]$. From the analyticity of the diagram and the fact that k^* is an isomorphism, it follows that $K[i^*] =$ $\delta'[H^{p-1}k(Y)] = \varphi'^{*-1}\delta f^* H^{p-1}(Y)$. Since φ'^* is an isomorphism, $H^p(Z(f,X),k(Y))/K[i^*]$ is isomorphic to $H^p(X,A)/$ $\delta f^*[H^{p-1}(Y)]$.

The exactness of the sequence $0 \longrightarrow H^0(X,A) \longrightarrow$ $H^0(Z(f,X)) \longrightarrow 0$ implies (ii).

Let E^n be the n-cell and let S^{n-1} be the boundary of E^n. If the coefficient group is the group of integers G, then $H^{n-1}(S^{n-1})$ is the group of integers. If f is a continuous function from S^{n-1} to S^{n-1}, then $\deg f = f^*(1)$ [6]. Thus, from the lemma and the fact that δ is an isomorphism if $X = E^n$ and $A = Y = S^{n-1}$, it follows that $H_n(Z(f,E^n))$ is isomorphic to $H^n(E^n,S^{n-1})/f^*H^{n-1}(S^{n-1})$, which is also isomorphic to $G/(\deg f)G$.

<u>1.2.11: Lemma.</u> If $H^{p-1}(X,A) = 0$, then $H^{p-1}(Z(f,X))$ is isomorphic to $K[\delta f^*]$ where δ is the co-boundary operator from $H^{p-1}(A)$ to $H^p(X,A)$.

Proof: In the following analytic diagram j is an inclusion map and the top row is exact.

$$\begin{tikzcd}
0 \arrow[r] & H^{p-1}(Z(f,X)) \arrow[r, "j"] & H^{p-1}(k(Y)) \arrow[r, "\delta'"] \arrow[d, "k^*"] & H^p(Z(f,X),k(Y)) \arrow[dd, "\varphi'^*"] \\
& & H^{p-1}(Y) \arrow[d, "f^*"] & \\
& & H^{p-1}(A) \arrow[r] & H^p(X,A)
\end{tikzcd}$$

Thus j^* is a monomorphism and $H^{p-1}(Z(f,X))$ is isomorphic to $K[\delta'] = K[\varphi'^{*-1}\delta f^* k^*]$, which is isomorphic to $K[\delta f^*]$ since φ^* and k^* are isomorphisms.

If $X = E^n$ and $Y = S^{n-1} = A$, then $H^{n-1}(Z(f,E^n))$ is isomorphic to $K[f^*] = \{g | g \in G \text{ and } (\deg f)g = 0\}$ since δ is an isomorphism. It follows that if f is a continuous function from S^{n-1} onto S^{n-1}, then $H^0(f,E^n)$ is isomorphic to G, $\{g | (\deg f)g = 0\}$, $G/(\deg f)G$, or 0, according as $p = 0$, $p = n-1$, $p = n$, or $p \neq 0, n-1, n$.

The following corollary is clear because of these computations.

<u>1.2.12: Corollary.</u> If f and g are continuous maps from S^{n-1} onto S^{n-1} and the coefficient groups are the group of integers, then $H^n(Z(f,E^n))$ is isomorphic to $H^n(Z(g,E^n))$ if and only if deg f = deg g.

<u>3; Elementary Propositions on the Point-Set Structure of Z(f,X).</u>

Let g be a continuous function from Y onto another compact Hausdorff space T. The first lemma in this section gives a relationship between $Z(gf,X)$ and $Z(gk^{-1}, Z(f,X))$.

<u>1.3.1: Lemma.</u> $Z(gf,X)$ is homeomorphic to $Z(gk^{-1}, Z(f,X))$.

Proof: Let φ' and φ'' be natural projections with ranges and domains as indicated below:

$$\begin{tikzcd}
X \arrow[rr, "\varphi''"] & & Z(gf,X) \\
\cup & & \\
A \arrow[rr, "gf"] & & T \arrow[uu, "k"']
\end{tikzcd}$$

$$\begin{tikzcd}
X \arrow[r, "\varphi"] & Z(f,X) \arrow[r, "\varphi"] & Z(gk^{-1},Z(f,X)) \\
\cup & \cup & \\
A \arrow[r, "kf"] & k(Y) \arrow[r, "gk^{-1}"] & T \arrow[uu, "k'"']
\end{tikzcd}$$

It will be shown that the equation $h(z) = \varphi' \varphi \varphi''^{-1}(z)$ defines a homeomorphism from $Z(gf,X)$ onto $Z(gk^{-1}, Z(f,X))$.

To see that h is well defined, let $x, y \in X$ such that $\varphi''(x) = \varphi''(y)$. This implies that $x = y$ or $gf(x) = gf(y)$. Thus if $x = y$, then it is clear that $\varphi'\varphi(x) = \varphi'\varphi(y)$, and if $gf(x) = gf(y)$, then $k'gk^{-1}kf(x) = k'gk^{-1}kf(y)$. But $k'gk^{-1} = \varphi'$ and $kf = \varphi$ so that $\varphi'\varphi(x) = \varphi'\varphi(y)$. It is clear that h is continuous since $h\varphi'' = \varphi'\varphi$ and the latter is continuous. The fact that h is onto is an immediate consequence of the fact that if $z \in Z(gk^{-1},Z(f,X))$, then there exists a point $x \in X$ such that $z = \varphi'\varphi\ (x) = h\ \varphi''(x)$. It remains only to show that h is one-to-one. If $x,y \in X$ such that $\varphi'\varphi(x) = \varphi'\varphi(y)$, then $x = y$ or $k''gk^{-1}kf(x) = k'gk^{-1}kf(y)$. If $x = y$, then $\varphi''(x) = \varphi''(y)$. If $k'gk^{-1}kf(x) = k''gk^{-1}kf(y)$, then $\varphi''(x) = gf(x) = gf(y) = \varphi''(y)$ since k' is one-to-one.

This lemma gives a way to break up the study of $Z(f,X)$ into two cases: (1) f is monotone and (2) f is

light. It is known that a continuous function f may be represented as the composition of a monotone continuous function g and a light continuous function g'. Thus, Z(f,X) is homeomorphic to Z(g',Z(g,X)) and Borsuk's Paste Job can be divided into a light part and into a monotone part. The difficulty is that even if A is a "very nice space", then g(A) need not be well-behaved and Z(g,X) can be even more pathological (See [5]). However, in case X is the two-cell, E^2 and A is the one-sphere S^1, then things are not too bad.

<u>1.3.2: Remark.</u> If g is a continuous monotone function defined on S^1, then $Z(g,E^2)$ is the two-cell or two-sphere.

Proof: Since g is monotone and continuous, $g(S^1)$ is a point, or S^1 and the natural projection φ' of E^2 onto $Z(g,E^2)$ is monotone [21]. If the continuous monotone image of E^2 is cyclic, then the image is a two-cell or a two-sphere [21, p. 173]. Thus, it is sufficient to show $Z(g,E^2)$ is cyclic. Suppose $Z(g,E^2)$ is not cyclic, i.e., there exists $z \in Z(g,E^2)$ such that z separates $Z(g,E^2)$, then there exist open disjoint sets K and T such that $K \cup T = Z(g,E^2) - \{z\}$. The sets $\varphi'^{-1}(T)$ and $\varphi'^{-1}(K)$ are open disjoint sets and $E^2 - \varphi'^{-1}(z) = \varphi'^{-1}(T) \cup \varphi'^{-1}(K)$. If $\varphi'^{-1}(z)$ is contained in the interior of E^2, then $\varphi'^{-1}(z)$ is a single point and since single points do not separate E^2, it follows that $\varphi^{-1}(z) \subset S^1$. However, subsets of S^1 do not separate E^2 and therefore $Z(g,E^2)$ must be cyclic.

If $Z(g,E^2)$ is a two-sphere, then the same theorem just quoted says $\varphi'(S^1)$ is a point and so Y must be a

point. Since this case is trivial, it can be said $Z(g,E^2)$ is a two-cell and $g(S^1)$ is S^1. Thus, it is enough to study the light maps defined on S^1 in order to determine the structure of $Z(g,E^2)$.

<u>1.3.3: Corollary.</u> If f is a continuous monotone function from S^1 into S^1, then deg f = 0, 1, or -1.

Proof: If f is not onto, then deg f = 0. The case of interest is when f is onto. If $Z(f,E^2)$ is a two-cell, then $Z(f,E^2)$ has the cohomology groups of a point and from the computation in Section 2 it is seen that deg f = 1.

The next question considered is: If f' is a continuous function from A onto a compact Hausdorff space Y', how "similar" does Y' have to be to Y to insure $H^p(Z(f,X))$ is isomorphic to $H^p(Z(f',X))$. Let $X = E^2$ and A = Y = Y' = S', then $H^p(Z(f,X))$ is isomorphic to $H^p(Z(f',X))$ if and only if deg f = $\pm$ deg f'. Thus, it is seen that in addition to restrictions on Y and Y' some restrictions must be placed on f and f'.

<u>1.3.4: Remark.</u> If g is a continuous function from Y to Y', if g* is an isomorphism, and if gf = f', then $H^p(Z(f,X))$ is isomorphic to $H^p(Z(f',X))$.

Proof: In the following analytic diagram φ' is the natural projection of X onto Z(f',X) and the function q from Z(f,X) to Z(f',X) is defined by $q(z) = \varphi'\varphi^{-1}(z)$. If f(x) = f(y), then f'(x) = gf(x) = gf(y) = f'(y) and so q is well defined. Since $q\varphi = \varphi'$, q is continuous.

$$\begin{array}{ccccccc}
X & \xrightarrow{\varphi} & Z(f,X) & \xrightarrow{q} & Z(f',X) & \xleftarrow{\varphi'} & X \\
\cup & & \uparrow k & & \uparrow k' & & \cup \\
A & \xrightarrow{f} & Y & \xrightarrow{g} & Y' & \xrightarrow{f'} & A
\end{array}$$

Define the function q' from $(Z(f,X),k(Y))$ to $(Z(f',X), k'(Y))$ by $q'(x) = q(z)$. It follows from the Map Excision Theorem that q'^* is an isomorphism. The function q'' from $k(Y)$ to $k'(Y)$ is defined by $q''(z) = q(z)$. Since $q'' = k'gk^{-1}$ and g^* is an isomorphism, q''^* is an isomorphism. Thus it follows from Lemma 1.1.2 that q^* is an isomorphism.

<u>1.3.5: Remark.</u> If g is a continuous function from Y to Y', if h is a continuous function from Y' to Y, if $gf = f'$, and if $hf' = f$, then $Z(f,X) = Z(f',X)$.

Proof: It is enough to show $R(f,X) = R(f',X)$.

If $(x,y) \in R(f,X)$, then $x = y$ or $f(x) = f(y)$. If $f(x) = f(y)$, then $f'(x) = gf(x) = gf(y) = f'(y)$. Similarly, $R(f',X)$ is a subset of $R(f,X)$.

CHAPTER II

HOMOTOPIC BORSUK'S PASTE JOBS HAVE THE SAME COHOMOLOGICAL STRUCTURE

The main result of this chapter is that if X and Y are compact Hausdorff spaces, if A is a closed subspace of X, and if f and g are homotopic functions from A onto Y, then $H^p(Z(f,X))$ is isomorphic to $H^p(Z(g,X))$ for all non-negative integers p.

By generalizing the definition of homotopic maps, it is possible to prove a more general theorem than indicated above. Section 0 contains the necessary definitions for this generalization. The proof of the theorem is divided into four parts, each of which constitutes a section in this chapter. Section 1 is devoted to a projection theorem; Section 2 contains a proof that if a Borsuk's Paste Job and a partial mapping cylinder have the same defining map, then they have isomorphic cohomology groups; Section 3 contains a technical proposition which shows that two particular partial mapping cylinders have isomorphic cohomology groups; and Section 4 combines the previous results to the main theorem.

0: T-Homotopy.

Conventions: For the remainder of the chapter,

T denotes a connected set. If H is a function from X x Y to Z and y is a point in Y, then h_y denotes the function from X to Z defined by $h_y(x) = H(x,y)$ for all x in X. The unit interval is denoted by I. All spaces are Hausdorff.

2.0.1: Definition. Let f and g be functions from X to Y and t and t' be points of T. Then f is T-homotopic to g at t, t' if and only if there exists a continuous function H from X x T to Y such that $h_t = f$ and $h_{t'} = g$. (If the points t and t' are not crucial, they will not be specified.) The space T will be called the connecting space.

This idea arises from the fact that the Homotopy Lemma [14] is nothing more than the Homotopy Theorem [14] stated for T-homotopic functions. The idea of T-homotopy has been used effectively in [15].

Some of the usual concepts involving I-homotopy can be introduced into this context.

2.0.2: Definition. A subspace A of a space X is a T-deformation retract of X if and only if there exists a retraction of X onto A such that the retraction is T-homotopic to the identity.

A space X is T-contractible to a point x from t on t' if and only if the injection of X onto X is T-homotopic at t, t' to the constant x-valued function on X.

2.0.3: Definition. Let P be a topological space.

(a) An operation (multiplication) defined on P is a continuous function m from P x P to P. The pair (P,m) is a topological algebra.

(b) Let m be an operation on P. Then a point p in P is a <u>left unit</u> for m if and only if $m(p,x) = x$ for all x in P. A point p in P is a <u>right unit</u> for m if and only if $m(x,p) = x$ for all x in P. A point p in P is a <u>(two-sided) unit</u> if and only if p is a left and a right unit for m.

(c) Let m be an operation on P. Then a point p in P is a <u>left zero</u> for m if and only if $m(p,x) = p$ for all x in P. A point p in P is a <u>right zero</u> for m if and only if $m(x,p) = p$ for all x in P. A point p in P is a <u>(two-sided) zero</u> if and only if p is a left and a right zero for m.

The connection between T-homotopy and topological algebra is quite strong. If T is a connected space and $t,t' \in T$, then T is T-contractible to t from t' on t if and only if T admits an operation m such that t' is a right unit for m and t is a zero for m. These types of conditions are needed in this chapter and these restrictions are phrased in the language of topological algebra.

Having made these definitions, the first question which arises is the question of how much of a generalization is T-homotopy over I-homotopy. The following observations, which were made by Professor A. D. Wallace, will serve to illustrate that T-homotopy is a proper generalization of I-homotopy.

<u>2.0.4: Observation.</u> If X is T-contractible, X is acyclic.

<u>2.0.5: Observation.</u> If T is a topological semi-lattice,

then T is T-contractible.

<u>2.0.6: Observation.</u> If X is I-contractible to a point x, X is arcwise connected at x to every other point in X.

<u>2.0.7: Observation.</u> The Long Line L [15] is L-contractible but not I-contractible.

Since the n-sphere is not acyclic, not every continuum is self-contractible.

For the remainder of the chapter T is a connected space and $t, t' \in T$.

The next two lemmas demonstrate that for a "large" class of spaces T, the ideas of T-homotopy and I-homotopy coincide.

A space P is <u>arcwise connected</u> between two points p and y of P if and only if there exists a homeomorphism h of I into P such that $h(0) = p$ and $h(1) = y$.

<u>2.0.8: Lemma.</u> If T is arcwise connected between t and t', if f and g are mappings from X to Y, and if f is T-homotopic to g at t,t', then f is I-homotopic to g.

Proof: There exists a continuous function H from $X \times T$ to Y such that $h_t = f$ and $h_{t'} = g$. Since T is arcwise connected between t and t', there exists a function h from I to T such that $h(0) = t$ and $h(1) = t'$. Let i be the injection of X onto X and define K from $X \times I$ to Y by $K = H(i \times h)$. Since $K(x,0) = H(x,h(0)) = H(x,t) = f(x)$ and $K(x,1) = H(x,h(1)) = h(x,t') = g(x)$ for all x in X, it follows that f is I-homotopic to g.

<u>2.0.9: Lemma.</u> If T is arcwise connected between t and t',

if T is a normal space, and if f is I-homotopic to g, then f is T-homotopic to g at t,t'.

Proof: There exists a continuous function h from I into T such that $h(0) = t$ and $h(1) = t'$. The Tietze Extension Theorem states h may be extended to a function h' whose domain is T.

Since f and g are I-homotopic, there exists a continuous function H from $X \times I$ to Y such that $h(x,0) = f(x)$ and $H(x,1) = g(x)$ for all x in X. Define K from $X \times T$ to Y by $K(x,t") = H(x,h'(t"))$. Clearly, K is continuous, $K(x,t) = H(x,h(t)) = H(x,0) = f(x)$, and $K(x,t') = g(x)$, so that f is T-homotopic to g.

These two lemmas say that if T is arcwise connected and normal, then f is T-homotopic to g if and only if f is I-homotopic to g.

Some properties of T-deformation retracts are studied in the remainder of this section.

<u>2.0.10: Lemma.</u> Let $0, 1 \in T$. If T is self-contractible to 0 from 1 on 0, then $T \times \{0\}$ is a T-deformation retract of $T \times T$. Also, $\{0\} \times T$ is a T-deformation retract of $T \times T$.

Proof: The hypothesis ensures the existence of a function H from $T \times T$ to T such that $h_1(t) = t$ and $h_0(t) = 0$ for all t in t. Define K from $(T \times T) \times T$ to $T \times T$ by $K((t_1,t_2),t_3) = (t_1,H(t_2,t_3))$. The function K is clearly continuous. Thus $k_1(t_1,t_2) = (t_1,h_1(t_2)) = (t_1,t_2)$ so that k_1 is the identity on $T \times T$. Since $k_0(t_1,t_2) = (t_1,h_0(t_2)) = (t_1,0)$, k_0 is a retraction of $T \times T$ onto $T \times \{0\}$.

It is well known that if X is a compact Hausdorff space and if A is a deformation retract of X, then the cohomology groups of X are isomorphic to those of A [14]. The same proof with minor changes will suffice to demonstrate the same result is true for T-deformation retracts.

<u>2.0.11: Lemma.</u> If A is a T-deformation retract of a compact Hausdorff space X and if i is the injection of A into X, then i^* is an isomorphism.

Proof: First it is shown that i^* is a monomorphism. If $g \in H^p(X)$ such that $i^*(g) = 0$, then there exists an open set M containing A such that $j^*(g) = 0$ [14, Reduction Theorem] where M^* denotes the closure of M and j is the injection of M^* into X. If r is the injection of X into (X,M^*), then $H^p(X,M^*) \xrightarrow{r^*} H^p(X) \xrightarrow{j^*} H^p(M^*)$ is exact so that there exists $g' \in H^p(X,M^*)$ such that $r^*(g') = g$.

Since A is a T-deformation retract of X, there exists a continuous function H from $X \times T$ to X and $t,t' \in T$ with the property that $h_{t'}$ is the identity function on X and h_t is a retraction of X onto A. Define the continuous function f from (X,M^*) into (X,X) by $f(x) = h_t(x)$. Since $H^p(X,X) = 0$ [14], it follows that f^* is identically zero.

If q is the injection of X into (X,X), then $fq(x) = h_t(x) = rh_t(x)$ which implies that $fq = rh_t$ and $q^*f^* = (fq)^* = (rh_t)^* = h_t{}^*r^*$. Thus $g = h_{t'}{}^*(g) = h_t{}^*(g) = h_t{}^*r^*(g') = q^*f^*(g') = 0$ and i^* is a monomorphism. Since A is a retract of X, i^* is an epimorphism [14, Proposition 53].

<u>2.0.12: Lemma.</u> If A is a T-deformation retract of X,

then $H^p(X,A)$ is trivial for all non-negative integers p.

Proof: Let j be the injection of X into (X,A) and i be the injection of A into X. Since i* is an isomorphism and

$0 \longrightarrow H^0(X,A) \xrightarrow{j^*} H^0(X) \xrightarrow{i^*} H^0(A)$ is exact, it follows that $0 = K[i^*] = I[j^*]$. Thus, the fact that j* is a monomorphism implies $H^0(X,A) = 0$.

Since $H^p(X) \xrightarrow{i^*} H^p(A) \xrightarrow{\delta} H^{p+1}(X,A) \xrightarrow{j^*} H^{p+1}(X) \xrightarrow{i^*}$ $H^{p+1}(A)$ is exact and i* is an epimorphism, δ is a zero map and j* is a monomorphism. But i* is a monomorphism so that j* is a zero map and hence $H^p(X,A) = 0$.

<u>2.0.13: Lemma.</u> If A is a T-deformation retract of X, then there exists a continuous function f' from X to X such that f' is T-homotopic to the identity mapping on X, $f'(X) = A$ and f' is idempotent. If the continuous function f from X to A is defined by $f(x) = f'(x)$, then the induced homomorphism f* is an isomorphism.

Proof: If i is the injection of A into X, then by Lemma 2.0.11 the induced homomorphism i* from $H^p(X)$ to $H^p(A)$ is an isomorphism.

Since if = f' implies f*i* = f'*, which is the identity on $H^p(X)$, it follows that f* is an isomorphism.

1. A Projection Theorem.

Notation: In the remainder of this chapter X and Y are compact Hausdorff spaces, A is a closed subspace of X, T is a continuum, and $\Sigma = (X \times \{t\}) \cup (A \times T)$. In this

section H is a continuous function from $A \times T$ onto Y.

2.1.1: Lemma. If h_t is onto Y, then $Z(H,\Sigma_t)$ is homeomorphic to $Z(h_t,X)$.

Proof: Let φ and φ' be the natural projections from Σ_t to $Z(H,\Sigma_t)$ and from X to $Z(h_t,X)$, respectively, and let θ be the restriction of φ to $X \times \{t\}$. Define the continuous function m from $X \times \{t\}$ to X by $m(x,t) = x$. It will be shown that the equation $k(z) = \varphi' m \theta^{-1}(z)$ defines a homeomorphism between $Z(H,\Sigma_t)$ and $Z(h_t,X)$.

To see that k is well defined, it should be noted that $\theta^{-1}(z)$ is not empty for all $z \in Z(H,\Sigma_t)$ and if $\theta(x,t) = \theta(y,t)$, then $x = y$ or $h_t(x) = H(x,t) = H(y,t) = h_t(y)$ so that $\varphi' m(x,t) = \varphi' m(y,t)$.

To prove that k is continuous, it is sufficient to show that $k^{-1}[B] = \theta m^{-1}\varphi'^{-1}[B]$, for if B is closed, then $\theta m^{-1}\varphi'^{-1}[B]$ is closed. If $z \in k^{-1}[B]$, then $\varphi' m \theta^{-1}(z) = k(z) \in B$ so that $z \in \theta m^{-1}\varphi'^{-1}[B]$. Conversely, if $z \in \theta m^{-1}\varphi'^{-1}[B]$, then $k(z) = \varphi' m \theta^{-1}(z) \in B$ and the equality follows.

In order to show that k is a one-to-one function, let $z,z' \in Z(H,\Sigma_t)$ such that $k(z) = k(z')$. Then there exist points $x,y \in X$ with the property that $\theta(x,t) = z$ and $\theta(y,t) = z'$. If $x \neq y$, then $x,y \in A$ and $\varphi'(a) = k(z) = k(z') = \varphi'(a)$ which implies that $H(x,t) = h_t(x) = h_t(y) = H(y,t)$, and therefore $z = \theta(x,t) = \theta(y,t) = z'$.

It remains only to show that k is onto. If $z \in Z(h_t,X)$, then there is $x \in X$ such that $z = \varphi'(x)$.

Since $k\varphi(x,t) = z$, k is onto.

2: The Cohomological Structures of Partial T-Mapping Cylinders and Borsuk's Paste Job are the Same.

Notation: Let 0 and 1 be two distinct points of T and let f be a continuous function from A onto Y.

2.2.1: Definition. Define the continuous function f' from $A \times \{1\}$ onto Y by $f'(x,1) = f(x)$. The partial mapping T-cylinder of f over X at 0,1 is the Borsuk's Paste Job $Z(f',\Sigma_o)$. Since this partial mapping T-cylinder is determined by f,X,0,1, and T, it is denoted by $M(f,X,0,1,T)$ (or by M if it is clear what f,X,0,1, and T are).

A partial mapping T-cylinder over X at 0,1 is called a partial mapping cylinder. This is the motivation for the definition which is given here. The reader is referred to [3] and [6] for more facts concerning partial mapping cylinders.

For the remainder of this chapter a standing hypothesis will be that there exists a continuous function H from $T \times T$ into T such that 0 is a zero for H and 1 is a unit for H. This assumption implies that T is T-contractible from 1 to 0.

2.2.2: Lemma. If p is a non-negative integer, then $H^p(X \times T, \Sigma_o) = 0$.

Proof: Define the continuous function K from $(X \times T \times T, \Sigma_o \times T)$ into $(X \times T, \Sigma_o)$ by $K(x,t_1,t_2) = (x,H(t,t_2))$. Since $k_1(x,t) = K(x,t,1) = (x,H(t,1)) = (x,t)$, k_1 is the

identity map on $(X \times T, \Sigma_o)$. But $k_o(x,t) = K(x,t,0) = (x,H(t,0)) = (x,0)$ so that $k_o(X \times T)$ is a subset of Σ_o. Thus $k_o{}^*$ is a zero function [14, Corollary 1]. Since k_o is T-homotopic to the identity map on (X, Σ_0), $k_o{}^*$ is the identity map on $H^p(X, \Sigma_o)$ and so $H^p(X, \Sigma_o) = 0$.

2.2.3: Lemma. If m is the continuous function defined on $X \times T$ to $X \times \{0\}$ by $m(x,t) = (x,0)$, then m^* is an isomorphism.

Proof: First it will be shown that $X \times \{0\}$ is a T-deformation retract of $X \times T$. The continuous function K which is defined by $K(x,t,t') = (x,H(t,t'))$ is a function from $X \times T \times T$ into $X \times T$. Since $k_1(x,t) = (x,t)$ and $k_o(x,t) = (x,0)$, it follows from Lemma 2.0.13 that m^* is an isomorphism.

2.2.4: Lemma. If v is a continuous function from Σ_o to $X \times \{0\}$ defined by $v(x,t) = (x,0)$ for all $(x,t) \in \Sigma_o$, then v^* is an isomorphism.

Proof: If r is defined by $r(x,t) = (x,t)$, then the following diagram is analytic:

$$\begin{tikzcd}
H^p(X \times T, \Sigma_o) \arrow[r] & H^p(X \times T) \arrow[r] & H^p(\Sigma_o) \\
H^p(X \times \{0\}, X \times \{0\}) \arrow[r] \arrow[u, "r^*"] & H^p(X \times \{0\}) \arrow[r] \arrow[u, "m^*"] & H^p(X \times \{0\}) \arrow[u, "v^*"']
\end{tikzcd}$$

It follows that r^* is an isomorphism since $H^p(X \times T, \Sigma_o) = 0 = H^p(X \times \{0\}, X \times \{0\})$. If m is as in Lemma 2.2.5, then m^* is an isomorphism. Thus, from Lemma 1.2.1, v^* is also an isomorphism.

<u>2.2.5: Lemma.</u> If the function u from $(\Sigma_o, A \times \{1\})$ to $(X \times \{0\}, A \times \{0\})$ is defined by $u(x,t) = (x,0)$, then u^* is an isomorphism.

Proof: If v is as in Lemma 2.2.4 and w is v restricted to $A \times \{1\}$, then the following diagram is analytic:

$$\begin{tikzcd} H^p(X \times \{0\}) \arrow[r] \arrow[d, "v^*"] & H^p(A \times \{0\}) \arrow[r] \arrow[d, "w^*"] & H^{p+1}(X \times \{0\}, A \times \{0\}) \arrow[d, "u^*"] \\ H^p(\Sigma_o) \arrow[r] & H^p(A \times \{1\}) \arrow[r] & H^{p+1}(\Sigma_o, A \times \{1\}) \end{tikzcd}$$

The mapping v^* is an isomorphism by Lemma 2.2.4 and w^* is also an isomorphism since w is a homeomorphism. It follows from Lemma 1.2.1 that u^* is an isomorphism.

No distinction will be made between X and $X \times \{0\}$ since they are topologically the same. Similarly, no distinction will be made between A and $A \times \{0\}$.

<u>2.2.6: Proposition.</u> The group $H^p(Z(f,X))$ is isomorphic to the group $H^p(M(f,X,0,1,T))$ for all non-negative integers p.

Proof: Let $Z = Z(f,X)$ and $M = M(f,X,0,1,T)$. In the following analytic diagrams φ_1 and φ_2 are the natural projections.

$$\begin{tikzcd} X \arrow[r, "\varphi_1"] & Z \\ A \arrow[u, phantom, "\cup"] \arrow[r, "f"] & Y \arrow[u, "k_1"'] \end{tikzcd} \qquad \begin{tikzcd} \Sigma_o \arrow[r, "\varphi_2"] & M \\ A \times \{1\} \arrow[u, phantom, "\cup"] \arrow[r, "f"] & Y \arrow[u, "k_2"'] \end{tikzcd}$$

The function h from M into Z is defined by $h(z) = \varphi_1 v \varphi_2^{-1}(z)$ where v is as in Lemma 2.2.4. It is sufficient to show that h^* is an isomorphism in order to establish the

proposition. First it is seen that h is well defined and continuous. If $\varphi_2(a,1) = \varphi_2(a',1)$ and $a \neq a'$, then $f(a) = f(a')$ so that $h\varphi_2(a,1) = \varphi_1 v(a,1) = \varphi_1(a,0) = \varphi_1(a,0) = \varphi_1(a',0) = \varphi_1 v(a',1) = h\varphi_2(a',1)$ and hence h is well defined. Since $h\varphi_2 = \varphi_1$, v is continuous, h is continuous.

Because $hk_2(Y)$ is a subset of $k_1(Y)$, the function p which is defined by $p(z) = h(z)$ is a mapping from $(M,k_2(Y))$ to $(Z,k_1(Y))$. If g denotes the restriction of h to $k_2(Y)$, then the following diagram is analytic:

$$\begin{array}{ccccc}
H^p(M,k_2(Y)) & \longrightarrow & H^p(M) & \longrightarrow & H^p(k_2(Y)) \\
\uparrow p^* & & \uparrow h^* & & \uparrow g^* \\
H^p(Z,k,(Y)) & \longrightarrow & H^p(Z) & \longrightarrow & H^p(k_1(Y))
\end{array}$$

In order to establish that h* is an isomorphism, it is enough to show that p* and g* are isomorphisms. Since $g\varphi_2(a,1) = \varphi_1 v\varphi_2^{-1}\varphi_2(a,1) = \varphi_1(a) = k_i f(a) = k_1 k_2^{-1}\varphi_2(a,1)$, $g = k_1 k_2^{-1}$ and so g* is an isomorphism since k_1^* and k_2^* are isomorphisms. Define the function φ_1' from (X,A) to $(X,k_1(Y))$ by $\varphi_1'(x) = \varphi(x)$ for all $x \in X$ and define the function φ_2' from $(\Sigma_o, A \times \{1\})$ to $(M,k_2(Y))$ by $\varphi_2'(x) = \varphi_2(x)$ for all $x \in \Sigma_o$. The Map Excision Theorem [13] says that $\varphi_1'^*$ and $\varphi_2'^*$ are isomorphisms. If u is defined as in 2.2.4, then $p\varphi_2'(x,t) = h\varphi_2(x,t) = \varphi_1 v(x,t) = \varphi_1' u(x,t)$ for all $(x,t) \in \Sigma_o$. Thus, $p\varphi_2' = \varphi_1' u$, and so $p^* = (\varphi_2'^*)^{-1} u^*(\varphi_1')^*$, which is an isomorphism.

3: A Relationship Between Two Partial Mapping T-Cylinders.

The purpose of this section is purely technical. A relation between two partial mapping T-cylinders is needed to establish the main theorem and it is in this section that this relationship is obtained.

For the remainder of this chapter it is assumed that there exists a continuous function K defined on $T \times T$ such that 1 is a zero for K and 0 is a unit for K.

Notation: Let P be a continuous function from $A \times T$ onto Y, let $M_o = M(P,\Sigma_o,0,1)$, let $M_1 = M(P,\Sigma_1,0,1)$, let $\Sigma_1' = (X \times \{1\} \times \{0\}) \cup (A \times T \times T) = (\Sigma_1 \times \{0\}) \cup (A \times T \times T)$, and let $\Sigma_o' = (X \times \{0\} \times \{0\}) \cup (A \times T \times T) = \Sigma_o \times \{0\} \cup (A \times T \times T)$. Let φ_o and φ_1 be the natural projections from Σ_o' to M_o and from Σ_1' to M_1, respectively. The mappings $\varphi_i'(0,1)$ from $(\Sigma_i', A \times T \times T)$ to (M_i, A_i') where $A_i = \varphi_i(A \times T \times T)$ are defined by $\varphi_i(x) = \varphi_i(x)$ where $i = 0,1$. It follows from the Map Excision Theorem [13] that $\varphi_1'^*$ and $\varphi_o'^*$ are isomorphisms. The purpose of this section is to prove that $H^p(M_o)$ is isomorphic to $H^p(M_1)$ for all non-negative integers p.

The function p from Σ_1' to Σ_o' is defined by $p(x,t,t') = (x,H(t,t'),t')$. It is clear that p is continuous and that $p(A \times T \times T)$ is contained in $A \times T \times T$. Hence, the continuous function j which is defined by $j(x,t,t') = p(x,t,t')$ is a mapping from $(\Sigma_1', A \times T \times T)$ to $(\Sigma_o', A \times T \times T)$.

<u>2.3.1: Lemma.</u> The mapping j^* is an isomorphism from $H^p(\Sigma_o', A \times T \times T)$ onto $H^p(\Sigma_1', A \times T \times T)$.

Proof: This lemma follows immediately from the Map Excision Theorem [13].

Let q be the function from M_1 to M_o defined by $q(z) = \varphi_o p \varphi^{-1}(z)$ for all $z \in M_1$. It will be shown that q^* is an isomorphism.

<u>2.3.2: Lemma.</u> The mapping q is well defined and continuous.

Proof: If $\varphi_1(a,t,1) = \varphi(a',t',1)$, then $P(a,t) = P(a',t')$ so that by using the definition of p and the properties of H, it is seen that $\varphi_o p(a,t,1) = \varphi_o(a,H(t,1),1) = \varphi_o(a,t,1) = \varphi_o(a',t',1) = \varphi_o(a',H(t',1),1) = \varphi_o p(a',t',1)$. Thus q is well defined.

Since $q\varphi_1 = \varphi_o p$, $q\varphi_1$ is continuous and it follows from Lemma 1.0.1 that q is also continuous.

Since $q(A_1')$ is contained in A_o', the continuous function r, defined by $r(z) = q(z)$, is a mapping from (M_1,A_1') to (M_o',A_o').

<u>2.3.3: Lemma.</u> The map r^* is an isomorphism.

Proof: From the definition of r, it is clear that $r\varphi_1' = \varphi_o' j$ so that $\varphi_1'^* r^* = (r\varphi_1')^* = (\varphi_o j)^* = j^* \varphi_o'^*$. Since $\varphi_1'^*$, j^*, and $\varphi_o'^*$ are isomorphisms, r^* is an isomorphism.

The function m from A_1' to A_o' is defined by $m(z) = q(z)$.

<u>2.3.4: Lemma.</u> The set $m(\varphi_1(A \times T \times \{1\}))$ is contained in $m(\varphi_o(A \times T \times \{1\}))$.

Proof: If $(a,t,1) \in (A \times T \times \{1\})$, then $m\varphi_1(a,t,1) = q\varphi_1(a,t,1) = \varphi_o p(a,t,1) = \varphi_o(a,H(t,1),1) = \varphi_o(a,t,1) \in m(\varphi_o(A \times T \times \{1\}))$.

For convenience, A_i'' will denote $m(\varphi_i(A \times T \times \{1\})$ where i is either 0 or 1. From Lemma 2.3.4 it is clear that it is possible to define a continuous function m' from (A_1', A_1'') to (A_o', A_o'') by $m'(z) = m(z)$ for all $z \in A_1'$. The map m", which is the restriction of m to A_1', is a continuous function from A_1' to A_o'.

<u>2.3.5: Lemma.</u> The mapping m" is a homeomorphism.

Proof: To see that m" is onto, let $z \in A_o''$. Then there exists $(a,t,1) \in A \times T \times \{1\}$ such that $\varphi_o(a,t,1) = z$. By using the appropriate definitions, it is seen that $m''\varphi_1(a,t,1) = q\varphi_1(a,t,1) = \varphi_o p(a,t,1) = \varphi_o(a,H(t,1),1) = \varphi_o(a,t,1) = z$.

It remains only to show that m" is a one-to-one function. If $m''\varphi_1(a,t,1) = m''\varphi_1(a',t',1)$, then by using the definitions of m, q, and p and by using the properties of H, it follows that $\varphi_o(a,t,1) = \varphi_o(a,H(t,1),1) = \varphi_o p(a,t,1) = q\varphi_1(a,t,1) = m''\varphi_1(a,t,1) = m''\varphi_1(a',t',1) = q\varphi_1(a',t',1) = \varphi_o p(a',t',1) = \varphi_o(a',t',1)$. Therefore, $P(a,t) = P(a',t')$ and thus $\varphi_1(a,t,1) = \varphi_1(a',t',1)$.

<u>2.3.6: Lemma.</u> The map m'^* is an isomorphism.

Proof: Let p' be the continuous function defined from $(A \times T \times T, A \times T \times \{1\})$ to $(A \times T \times T, A \times T \times \{1\})$ by $p'(a,t,t') = (a,H(t,t'),t')$. Define the continuous function L from $(A \times T \times T \times T, A \times T \times \{1\} \times T)$ to

$(A \times T \times T, A \times T \times \{1\})$ by $L(a,t,t',t'') = (a,H(t,K(t',t'')), t')$. Since $L(a,t,1,t') = (a,H(t,K(1,t')),1)$, the domain and range are correct. Since $L(a,t,t',1) = (a,H(t,K(t',1)), t') = (a,H(t,1),t') = (a,t,t')$ and $L(a,t,t',0) = (a,H(t,K(t',0)),t') = (a,H(t,t'),t') = p'(a,t,t')$, p' is T-homotopic to the identity on $(A \times T \times T, A \times T \times \{1\})$. Thus p'^* is an isomorphism.

Let φ_1'' and φ_o'' be the continuous functions from $(A \times T \times T, A \times T \times \{1\})$ to (A_o', A_o'') and (A_1',A_1''), respectively, which are defined by $\varphi_i''(x) = \varphi_i(x)$, for $i = 0,1$. It follows immediately from the Map Excision Theorem that $\varphi_1''^*$ and $\varphi_o''^*$ are isomorphisms.

The following diagram is analytic :

$$\begin{array}{ccc} H^p(A_o',A_o'') & \xrightarrow{\varphi_o''^*} & H^p(A \times T \times T, A \times T \times \{1\}) \\ \uparrow m'^* & & \uparrow p'^* \\ H^p(A_1',A_1'') & \xrightarrow{\varphi_1^*} & H^p(A \times T \times T, A \times T \times \{1\}) \end{array}$$

It is necessary to show that $m'\varphi_1'' = \varphi_o''p'$. Clearly, $m'\varphi_1''$ and $\varphi_o''p'$ have the same range and domain. Since $m'\varphi_1''(a,t,t') = \varphi_o p(a,t,t') = \varphi_o''(a,H(t,t'),t') = \varphi_o''p'(a,t,t')$, $\varphi_1''m'^* = p'^*\varphi_o''$ and hence m'^* is an isomorphism because $p'^*, \varphi_o''^*$, and $\varphi_1''^*$ are isomorphisms.

<u>2.3.7: Lemma.</u> The mapping m^* is an isomorphism.

Proof: Consider the following diagram:

$$\begin{tikzcd}
H^p(A_1', A_1'') \arrow[r] & H^p(A_1') \arrow[r] & H^p(A_1'') \\
H^p(A_o', A_1'') \arrow[r] \arrow[u, "m''*"] & H^p(A_o') \arrow[r] \arrow[u, "m*"] & H^p(A_o') \arrow[u, "m''*"]
\end{tikzcd}$$

where the unnamed mappings are the usual injections. The mappings m"* and m'* are isomorphisms so that it follows from Lemma 1.2.1 that m* is also an isomorphism.

<u>2.3.8: Lemma.</u> The mapping q* is an isomorphism.

Proof: In the following diagram the unnamed mappings are injections.

$$\begin{tikzcd}
H^p(M_o, A_o') \arrow[r] & H^p(M_o) \arrow[r] & H^p(A_o') \\
H^p(M_1, A_1') \arrow[r] \arrow[u, "j*"] & H^p(M_1) \arrow[r] \arrow[u, "q*"] & H^p(A_1') \arrow[u, "m*"]
\end{tikzcd}$$

From Lemma 2.3.3, r* is an isomorphism and from Lemma 2.3.7, m* is an isomorphism. It follows that q* is also an isomorphism.

To summarize the results of this section, the following proposition is stated.

<u>2.3.9: Proposition.</u> Let T be a continuum and let $0, 1 \in T$ such that T admits a multiplication for which 0 is a zero and 1 is a unit and such that T admits a multiplication for which 1 is a zero and 0 is a unit. Let X and Y be compact Hausdorff spaces and let A be a closed subset of X. If P is a continuous function from $A \times T$ onto Y, then $H^p(M(\Sigma_o, P, 0, 1, T))$ is isomorphic to $H^p(M(\Sigma_1, P, 0, 1, T))$.

4: The Main Theorem.

In essence, the main theorem has already been proved; it remains only to state it and tie the loose ends together.

2.4.1: Proposition. Let X and Y be compact Hausdorff spaces, let A be a closed subset of X, and let T be a continuum. If f and g are continuous functions from A onto Y which are T-homotopic at 0,1, and if T admits two operations H and K such that 0 is a zero for H and a unit for K and such that 1 is a zero for K and a unit for H, then $H^p(Z(f,X))$ is isomorphic to $H^p(Z(g,X))$ for all non-negative integers p.

Proof: Since f is T-homotopic to g at 0,1, there exists a continuous function P from A x T onto Y such that $P(x,0) = f(x)$ and $P(x,1) = g(x)$ for all x in A.

From Lemma 2.1.1, it follows that $H^p(Z(P,\Sigma_o))$ is isomorphic to $H^p(Z(f,X))$ and $H^p(Z(P,\Sigma_1))$ is isomorphic to $H^p(Z(g,X))$. From Proposition 2.2.5, $H^p(Z(P,\Sigma_1))$ is isomorphic to $H^p(M(P,\Sigma_1,0,1,T))$ and $H^p(Z(P,\Sigma_o))$ is isomorphic to $H^p(M(P,\Sigma_o,0,1,T))$. Therefore, by Proposition 2.3.9, $H^p(M(P,\Sigma_o,0,1,T))$ is isomorphic to $H^p(M(P,\Sigma_1,0,T))$ and hence $H^p(Z(f,X))$ is isomorphic to $H^p(Z(g,X))$.

The hypotheses of Proposition 2.4.1 are clearly satisfied by any topological lattice; in particular, these hypotheses are satisfied by the unit interval. Thus, it follows from Lemma 2.0.8 that if $T = S^1$, then the conclusion of Proposition 2.4.1 holds even though the hypotheses of Proposition 2.4.1 are not satisfied by S^1. To avoid this

type of example, Proposition 2.4.1 is re-stated in a slightly different form.

<u>2.4.2: Theorem.</u> Let X, Y, and A be as in Proposition 2.4.1 and T' be a continuum. If f and g are continuous functions from A onto Y such that f is T'-homotopic to g at 0, 1, and if there exists a subcontinuum T of T; with 0, 1 ϵ T, and if T satisfies the hypotheses of Proposition 2.4.1, then $H^p(Z(f,X))$ is isomorphic to $H^p(Z(g,X))$ for all non-negative integers p.

Proof: There exists a continuous function P from A $\times$ T' onto Y such that $P(x,0) = f(x)$ and $P(x,1) = g(x)$ for all x in A. Since restriction of P to A $\times$ T demonstrates the fact that f is T-homotopic to g at 0, 1, the conclusion of the theorem follows immediately from Proposition 2.4.1.

CHAPTER III

BORSUK'S PASTE JOB IN SEMIGROUPS

If S is a semigroup, if A is a closed subsemigroup of S, and if f is a continuous epimorphism defined on A, then the question arises as to whether Z(f,S) admits a natural semigroup structure. To be more explicit, is R(f,S) a congruence of S? The answer is sometimes. Thus, the problem which is studied in this chapter is to find conditions on S and A which will insure that R(f,S) is a congruence of S for all epimorphisms f defined on A. Two propositions of this type will be proved in Section 2. In Section 1 there is a more technical discussion of the problem and an indication of why certain hypotheses are necessary.

A simple example which is contained in Section 3 shows that there exists a semigroup X and an epimorphism g defined on the minimal ideal of X such that R(g,X) is not a congruence of X. Since this example consists of left zeros, the case in which the subsemigroup A consists of left zeros is investigated in Section 4. The main result of these investigations is a theorem which gives a set of necessary and sufficient conditions for R(f,S) to be a congruence of a semigroup S for all epimorphisms f defined

on a subsemigroup A of S in the case that A consists of left zeros. In Section 5 this theorem is extended to obtain a set of necessary and sufficient conditions for $R(f,S)$ to be a congruence for all epimorphisms f defined on the minimal ideal.

In Section 6 there is presented a more general version of the theorems of Sections 4 and 5. The theorem in Section 6 gives necessary and sufficient conditions for $R(f,S)$ to be a congruence of S for all epimorphisms defined on a subsemigroup A. This characterization is in terms of the congruences of A.

0: Notation.

A semigroup is a non-empty Hausdorff space together with a continuous, associative multiplication. Precisely, a semigroup is a non-empty Hausdorff space S and a function m from $S \times S$ to S that satisfies the following conditions:

(1) S is a non-empty Hausdorff space,

(2) m is continuous,

(3) m is associative, i.e.,

$m(x,m(y,z)) = m(m(x,y),z)$ for all $x,y,z \in S$.

For convenience, $m(x,y)$ will be denoted by xy and following customary usage, we shall say "S is a semigroup" when it is clear what the multiplication for S is. In this chapter S is a semigroup.

If A and B are subsets of S, then $AB = \{ab | a \in A$ and $b \in B\}$. A non-empty subset L of S is a left ideal of S if and only if SL is a subset of L. A non-empty subset R of S

is a <u>right ideal</u> of S if and only if RS is a subset of R. A non-empty subset I of S is a (two-sided) <u>ideal</u> of S if and only if $SI \cup IS$ is a subset of I. A compact semigroup has minimal left, right, and two-sided ideals. The minimal ideal, if it exists, is unique and is denoted by K.

A point e of S is said to be <u>idempotent</u> if and only if $e^2 = ee = e$. Let E be the set of idempotents of S. It is known that if $e \in E$, then there exists a maximal subgroup H_e of S which contains e. If $e \in E \cap K$, then $eSe = H_e$, eS is a minimal right ideal, Se is a minimal left ideal, and $eS \cap Se = eSe$. Also, $K = \cup\{eSe | e \in E \cap K\} = \cup\{R | R$ is a minimal right ideal of $S\} = \cup\{L | L$ is a minimal left ideal of $S\}$. See [2], [10], and [15].

1: Preliminary Propositions.

<u>3.1.1: Definition.</u> If S is a semigroup and if F is a closed equivalence relation defined on S, then F is a <u>congruence</u> of S if and only if $(\Delta S)F \cup F(\Delta S)$ is a subset of F, where ΔS is the diagonal of $S \times S$.

The following lemma is well known [15].

<u>3.1.2: Lemma.</u> Let S be a compact semigroup and let F be a closed equivalence relation defined on S. Then F is a congruence of S if and only if S/F admits a unique continuous, associative multiplication m* such that $m^*(\varphi \times \varphi) = m\varphi$ where φ is the natural projection of S onto S/F and m is the multiplication defined on S.

$$\begin{tikzcd}
S/F \times S/F \arrow[rr, "m^*"] & & S/F \\
S \times S \arrow[u, "\varphi \times \varphi"'] \arrow[rr, "m"] & & S \arrow[u, "\varphi"']
\end{tikzcd}$$

The set S/F is said to admit a <u>desirable</u> multiplication if and only if F is a congruence of S.

Let S and T be compact semigroups, let A be a closed subset of S, and let f be a continuous function defined on A. The problem studied in this chapter is to determine when $R(f,S)$ is a congruence of S. The following lemma will give a criterion which is useful in studying this problem.

<u>3.1.3: Lemma.</u> Let S and T be compact semigroups, let A be a closed subset of S, and let f be a continuous function from A onto T. Then $R(f,S)$ is a congruence of S if and only if (3.1) is satisfied.

(3.1) If $x \in S$, if $a, a' \in A$, and if $f(a) = f(a')$, then $f(xa) = f(xa')$ or $xa = xa'$, and $f(ax) = f(a'x)$ or $ax = a'x$.

Proof: The necessity is obvious.

Sufficiency: If $x \in S$ and $(y,z) \in R(f,S)$, then $f(y) = f(z)$ or $y = z$. If $y = z$, then $xy = xz$ and $yx = zy$ so that (xy, xz) and (yx, zy) are in $R(f,S)$. If $f(x) = f(z)$, then it follows from (3.1) that (yx, zx) and (xy, xz) are in $R(f,S)$.

<u>3.1.4: Remark.</u> If S, A, T, and f are as in Lemma 3.1.3, and if f is a one-to-one function, then $R(f,S)$ is a congruence of S.

<u>3.1.5: Lemma.</u> If S, A, T, and f are as in Lemma 3.1.3, if φ is the natural projection, if R(f,S) is a congruence, and if k is the unique homeomorphism such that kf is the restriction of φ to A, then $kf(xy) = kf(x)kf(y)$ whenever $x,y,xy \in A$.

$$\begin{tikzcd}
S \arrow[r, "\varphi"] \arrow[d, phantom, "\cup"] & Z(f,S) \\
A \arrow[r, "f"] & T \arrow[u, "k"']
\end{tikzcd}$$

<u>3.1.6: Remark.</u> If S, A, T, f, φ, k, and R(f,S) are as in Lemma 3.1.5, then k is an is iseomorphism (both an isomorphism and a homeomorphism) if and only if f is a homomorphism and A is a subsemigroup of S.

Proof: Assume A is a subsemigroup of S and suppose that f is a homomorphism. If $y, y' \in T$, then there exist x, $x' \in A$ such that $y = f(x)$ and $y' = f(x')$. It follows that $k(y)k(y') = k(f(x))k(f(x')) = kf(xx') = k(f(x)f(x")) = k(yy')$ and thus k is homomorphism. Since k is a homeomorphism, k is an iseomorphism.

If k is an iseomorphism, then $k(T) = \varphi(A)$ is a subsemigroup and so $A = \varphi^{-1}\varphi(A)$ is a subsemigroup of S. Also, if $x,y \in A$, then $f(x)f(y) = k^{-1}\varphi(x)k^{-1}\varphi(y) = k^{-1}(\varphi(x)\varphi(y)) = k^{-1}\varphi(xy) = f(xy)$ since k^{-1} and φ are homomorphisms.

Let S and T be compact semigroups, let A be a closed subset of S, and let f be a continuous function from A onto T. If R(f,S) is a congruence of S, then it is desirable to be able to embed T iseomorphically into Z(f,S) by a function k such that $k^{-1}\varphi = f$, where φ is the natural projection of

S onto $Z(f,S)$ restricted to A. In view of the above remarks and Remark 3.1.6, we shall restrict our attention to the case when A is a subsemigroup of S and f is a homomorphism. It will be shown that it is also necessary to restrict our attention more by requiring A to be similar to an ideal. This is the reason for the next definition.

Let S be a semigroup and let B be a subset of S. Then $B^{(-1)}B = \{x \in S \mid Bx \cap B \neq \emptyset\}$, $BB^{(-1)} = \{x \in S \mid xB \cap B \neq \emptyset\}$, $B^{[-1]}B = \{x \in S \mid Bx \subset B\}$, and $BB^{[-1]} = \{x \in S \mid xB \subset B\}$.

<u>3.1.7: Definition.</u> Let S be a semigroup and let A be a non-empty subset of S, then A is a <u>semi-ideal</u> of S if and only if

(i) A is closed,

(ii) If $x \in S$, then $\operatorname{card}[xA \cap (S \setminus A)] \leq 1$ and $\operatorname{card}[Ax \cap (S \setminus A)] \leq 1$, and

(iii) $A^{(-1)}A \subset A^{[-1]}A$ and $AA^{(-1)} \subset AA^{[-1]}$.

The following remark gives a few of the properties of semi-ideals.

<u>3.1.8: Remark.</u> (a) Let A be a closed subset of a semigroup S, then A is a semi-ideal of S if and only if

(i) If $x \in S$, then $\operatorname{card} xA = 1$ or $x \in AA^{[-1]}$, and

(ii) If $x \in S$, then $\operatorname{card} A = 1$ or $x \in A^{[-1]}A$.

(b) A closed ideal of S is a semi-ideal of S.

(c) If A is a semi-ideal of S and if A contains an ideal of S, then A is an ideal of S.

(d) If G is a topological group and if A is a closed subset of G, then A is a semi-ideal of G if and only if card $A = 1$ or $A = G$.

(e) The intersection of a filter base of semi-ideals of a compact semigroup is a semi-ideal of that semigroup.

(f) If B is a subset of a compact semigroup, then there exists a semi-ideal minimal with respect to containing B.

(g) Semi-ideals are preserved under closed epimorphisms.

(h) Semi-ideals are not necessarily preserved under inverses of continuous epimorphisms.

Proof: (a) If A is a semi-ideal of S and if $x \in S$, then $x \in (S \setminus AA^{(-1)})$ or $x \in AA^{[-1]}$. If $x \in (S \setminus AA^{(-1)})$, then $xA \subset (S \setminus A)$ so that card $xA = \text{card}[xA \cap (S \setminus A)] = 1$. The proof of (ii) follows in a similar fashion to that of the proof of (i).

Suppose (i) and (ii) hold and let $x \in AA^{(-1)}$. It follows that $x \in AA^{[-1]}$ or card $xA = 1$. If card $xA = 1$, then $x \in AA^{[-1]}$ so that $AA^{(-1)}$ is a subset of $AA^{[-1]}$. It is clear that the other parts of the definition of a semi-ideal are also satisfied.

(b) is clear.

(c) If $I \subset A$, if A is a semi-ideal of S,

if I is an ideal of S, and if $x \in S$, then $\square \neq I \cap Ix \subset A \cap AX$ so that $x \in AA^{[-1]}$ and A is a right ideal. Similarly, A is a left ideal.

(d) The sufficiency is clear.

Necessity: If card $A > 1$, then there exists a point x in A such that x is distinct from the identity e of G. Since card $x^{-1}A$ is greater than 1, $x^{-1}A$ is a subset of A and so e is an element of A. If $y \in G$, then the cardinality of yA is greater than one. Hence $y \in yA \subset A$ and therefore $A = G$.

(e) Let $\mathfrak{B}$ be a filter base of semi-ideals of a compact semigroup S, then $B = \cap\{B' | B' \in \mathfrak{B}\}$ is closed.

If $x \in BB^{(-1)}$, then $\square \neq xB \cap B \subset xB' \cap B'$ for all $B' \in \mathfrak{B}$. Since each B' is a semi-ideal, $x \in B'B'^{[-1]}$ so that $xB \subset xB' \subset B'$. Thus, $xB \subset \cap\{B' | B' \in \mathfrak{B}\} = B$ so that $BB^{(-1)} \subset BB^{[-1]}$.

If $x \in S - BB^{(-1)}$, then there exists $B_o \in \mathfrak{B}$ such that $xB_o \cap B_o = \square$. For suppose $xB' \cap B' \neq \square$ for all $B' \in \mathfrak{B}$, then $\square \neq \{xB' \cap B' | B' \in \mathfrak{B}\} = (\cap\{xB' | B' \in \mathfrak{B}\}) \cap (\cap\{B' | B \in \mathfrak{B}\}) \subset xB \cap B$ so that $x \in BB^{(-1)}$ which is a contradiction.

Right multiplication follows in a similar fashion.

A straight-forward application of the Hausdorff Maximality Principle will prove (f).

(g) Let S be a semigroup, let f be a closed eipmorphism from S onto a semigroup T, and let A be a semi-ideal of S.

If $f(x) \in (f(A))(f(A)^{(-1)})$, then $xA \subset A$ or card $xA = 1$. If $xA \subset A$, then $f(x)f(A) = f(xA) \subset f(A)$ so that $f(x) \in f(A)(f(A)^{[-1]})$. If card $xA = 1$, then card $f(x)f(A) = 1$ and $f(x) \in f(A)(f(A)^{[-1]})$.

Suppose $f(x) \in (T - [(f(A))(f(A)^{[-1]})])$, then $f(xA \cap A) \subset f(xA) \cap f(A) = \square$ so that $x \in (S - AA^{(-1)})$. This means that card $xA = 1$ and therefore card $f(x)f(A) = 1$.

Right multiplication is verified in a similar fashion.

(h) Let S and T be the semigroup of complex numbers whose modulus is one under the usual multiplication. Define f from S to T by $f(z) = z^2$. It follows from (d) that each point w of T is a semi-ideal, but $f^{-1}(w)$ is not a semi-ideal.

The following lemma is a re-statement of Lemma 3.1.3.

<u>3.1.9: Lemma</u>. Let S and T be compact semigroups, let A be a subsemigroup and semi-ideal of S, and let f be a continuous epimorphism from A onto T. Then $R(f,S)$ is a congruence of S if and only if

(3.1.9.1) $x \in [(AA^{[-1]}) - A]$ and $f(a) = f(a')$ imply $f(xa') = f(xa)$ and

(3.1.9.2) $x \in [(A^{[-1]}A)]$ and $f(a) = f(a')$ imply $f(ax) = f(a'x)$.

Proof: Sufficiency. It is enough to verify 3.1. If $x \in S$, then $x \in A$ or $x \in [S - (AA^{[-1]})]$ or $x \in [(AA^{[-1]}) - A]$. Suppose $a,a' \in A$ and $f(a) = f(a')$, then if $x \in A$ it follows that $f(xa) = f(x)f(a) = f(x)f(a') = f(xa')$. If $x \in (S - (A^{[-1]}A))$, then $xa = xa'$ since card $xA = 1$. If

$x \in [(AA^{[-1]}) - A]$, it follows from 3.1.9.1 that $f(xa) = f(xa')$. The other part of 3.6 follows in similar fashion.

The necessity is obvious.

This lemma makes explicit the following idea. If A is a subsemigroup and a semi-ideal of S, then S is partitioned into three sets: $A, [(A^{[-1]}A) - A]$ and $[S - (A^{(-1)}A)]$. It is easy to see that if x is in $A \cup (S - A^{(-1)}A)$, then $(x,x) \cdot Z(f,S)$ is a subset of $Z(f,S)$. In order to say that $Z(f,S)$ is a congruence, one must, however, know what the behavior is of the points in $[(A^{[-1]}A) - A]$. Similar statements can be made about right multiplication.

Semi-ideals arose out of the necessity of having a subset satisfy conditions similar to the conditions for being an ideal. It will be shown that the property of being a semi-ideal is the "weakest" assumption that will work for the theorems presented here.

<u>3.1.10: Definition.</u> Let A be a closed subsemigroup of a compact semigroup S, let T be a degenerate semigroup, and let f be a continuous epimorphism from A onto T. The space $Z(f,S)$ is called the <u>Generalized Rees Quotient</u> of S by A and is denoted by $Q(A,S)$. Equivalently, $Q(A,S) = S/R$ where $R = (A \times A) \cup (\Delta S)$. The Generalized Rees Quotient is an extreme case of Borsuk's Paste Job and as such is helpful in determining restrictions which should be placed on A.

<u>3.1.11: Proposition.</u> Let A be a closed subsemigroup of a compact semigroup S. Then $Q(A,S)$ admits a desirable

multiplication (i.e., $(A \times A) \cup (\Delta S)$ is a congruence of S) if and only if A is a semi-ideal of S.

Proof: Let T and f be as in Definition 3.1.10.

Necessity. Suppose $x \in AA^{(-1)}$, then there exists a point y in A such that $xy \in xA \cap A$, then $xy \neq xz$. This implies that (3.1) is not satisfied, which is a contradiction. Thus $AA^{(-1)}$ is a subset of $AA^{[-1]}$.

Suppose $x \in S$ and $y,x \in A$ such that $xy, xz \in (S - A)$. According to (3.1), $xy = xz$ so that the cardinality of $xA \cap (S - A)$ is no greater than 1.

Right multiplication may be verified in a similar fashion.

Sufficiency. It is enough to verify 3.1.9.1 and 3.1.9.2. If $x \in [(AA^{[-1]}) - A]$ and $a,a' \in A$, then $f(xa) = f(xa')$ since the range of f is degenerate. The property 3.1.9.2 may be verified in a similar fashion.

<u>3.1.12: Definition.</u> If A is a compact subsemigroup of a compact semigroup S, then A is <u>agreeable</u> with respect to S if and only if R(f,S) is a congruence of S for every continuous epimorphism f defined on A.

Thus it is now possible to restate the problem which is studied in this chapter in technical language. The problem is to find conditions which will insure that a subsemigroup is agreeable.

<u>3.1.13: Remark.</u> (a) If S is a compact semigroup and if a is an idempotent, then $\{a\}$ is agreeable with respect to S.

(b) If S is a compact semigroup, then S is agreeable with respect to S.

<u>3.1.14: Remark.</u> If A is a closed subsemigroup of a compact semigroup S, then A is agreeable with respect to S if and only if $F \cup (\Delta S)$ is a congruence of S for every congruence F of S.

Proof: Necessity. Suppose F is a closed congruence of A. If f is the natural projection of A onto A/F, then $R(f,S) = (F \cup \Delta S)$ is a congruence of S since A is agreeable with respect to S.

Sufficiency. If f is a continuous epimorphism from A onto T, then $[(f \times f)^{-1}(\Delta T)] \cup \Delta S = R(f,S)$, and since $[(f \times f)^{-1}(\Delta T)]$ is a congruence of S, then $R(f,S)$ is a congruence of S.

<u>3.1.15: Proposition.</u> Let A be a closed subsemigroup of a compact semigroup S. If A is agreeable with respect to S, then every semi-ideal of A is a semi-ideal of S.

Proof: If P is a semi-ideal of A, then the natural projection f of A onto $Q(P,S)$ is a continuous epimorphism. Since A is agreeable with respect to S, $Z(f,S) = Q(P,S)$ admits a desirable multiplication and it follows from Proposition 3.1.11 that P is a semi-ideal of S.

It will be shown later that the converse to this proposition is not true. However, the converse is true in a certain special case which will be investigated in Section 4.

2: Sufficient Conditions.

3.2.1: Definition. Let S be a semigroup and let r be a function from S to S. Then r is a homomorphic retraction if and only if r is a homomorphism and a retraction, and r(S) is a homomorphic retract of S whenever r is a homomorphic retraction of S.

In the remainder of this section S is a semigroup and A is a subsemigroup and semi-ideal of S.

3.2.2: Lemma. The sets $AA^{[-1]}$ and $A^{[-1]}A$ are subsemigroups of S which contain A as a left ideal and as a right ideal, respectively.

Proof: If $x,y \in AA^{[-1]}$, then $xyA = x(yA) \subset xA \subset A$ so that $AA^{[-1]}$ is a subsemigroup of S and A is a left ideal of $AA^{[-1]}$. The proof for $A^{[-1]}A$ is similar.

3.2.3: Proposition. If A is a homomorphic retract of $AA^{[-1]}$ and $A^{[-1]}A$, then A is agreeable with respect to S.

Proof: It is enough to show that 3.1.9 is satisfied for an arbitrary continuous epimorphism f defined on A.

Let r be a homomorphic retraction from $A^{[-1]}A$ onto A. If $x \in [(A^{[-1]}A) - A]$ and $a,a' \in A$ such that $f(a) = f(a')$, then, using the properties of a homomorphic retraction and the fact that A is an ideal in $A^{[-1]}A$, it is seen that $f(ax) = f(r(ax)) = f(r(a)r(x)) = f(ar(x)) = f(a)f(r(x)) = f(a')f(r(x)) = f(a'r(x)) = f(r(a')r(x)) = f(r(a'x)) = f(a'x)$. The property 3.1.9.1 follows in a similar fashion.

Anticipating the results of Chapter IV, it may be seen that conditions on A, such as A contains a unit, will

insure that A is a homomorphic retract of $A^{[-1]}A$ and $AA^{[-1]}$. Using this result, the following corollary is obtained.

3.2.4: Corollary. If the minimal ideal of S is a group, then the minimal ideal is agreeable with respect to S.

The following proposition will be useful in Sections 4 and 5.

3.2.5: Proposition. If A satisfies the following conditions, then A is agreeable with respect to S.

(1) If $x \in [(AA^{[-1]}) - A]$, then card $xA = 1$ or $xa = a$ for all a in A.

(2) If $x \in [(A^{[-1]}A) - A]$, then card $Ax = 1$ or $ax = a$ for all a in A.

Proof: It is enough to verify 3.1.9 for an arbitrary epimorphism f defined on A.

If $x \in [(A^{[-1]}A) - A]$ and if $a, a' \in A$ such that $f(a) = f(a')$, then card $Ax = 1$, or $ax = a$, and $a'x = a'$. If card $Ax = 1$, then $ax = a'x$. If $ax = a$ and $a'x = a'$, then $f(ax) = f(a'x)$. The verification of 3.1.9.1 is similar.

3: Examples.

In this section, S will always be a semigroup.

The following lemma will be useful in this section.

3.3.1: Lemma. (a) If A is a closed subset of S and if S consists of left zeros, then A is a semi-ideal of S.

(b) If A is a closed subset of S and S consists of right zeros, then A is semi-ideal of S.

Proof: If $x \in S$, then $xA = x$ so that the cardinality of

xA is one. Since $Ax = A$, $A^{[-1]}A = S$.

The proof of (b) is similar to that of (a).

<u>3.3.2: Example.</u> This is an example of a semigroup with a semi-ideal which is not an ideal but which is agreeable.

Let S be a compact Hausdorff space with the multiplication defined by $xy = x$ and let A be a proper subset of S. Since A is a proper subset of S, there exists a point z in S which is not in A so that $zA = z$, which means A is not an ideal.

It follows from Lemma 3.3.1 that A is a semi-ideal and it follows from Proposition 3.2.5 that A is agreeable.

<u>3.3.3: Example.</u> This is an example of a semigroup whose minimal ideal satisfies the hypotheses of Proposition 3.2.5.

Let J be the unit interval with the multiplication defined by $xy = x$, let P be $\{0,1\}$ with the usual multiplication, and let S be $J \times P$ with the co-ordinatewise multiplication. The minimal ideal of S is $J \times \{0\}$. It is easily verified that the hypotheses of Proposition 3.2.5 are satisfied.

The following example is due to P. E. Connor.

<u>3.3.4: Example.</u> Let X be the unit interval with the usual multiplication, let Y be the space of complex numbers with norm one endowed with the usual multiplication, let S be $X \times Y$ endowed with co-ordinatewise multiplication, and let f be the continuous function from $\{0\} \times Y$ to Y defined by $f(0,z) = z^2$. The relation $R(f,S)$ is a congruence of S and $Z(f,S)$ is topologically a Möbius Band.

<u>3.3.5: Example.</u> This is an example of a semigroup whose minimal ideal is not agreeable.

Let S be the triangle semigroup, i.e., $S = \left\{ \begin{bmatrix} x & y \\ 0 & 1 \end{bmatrix} \middle| \, x \right.$ and y are real non-negative numbers such that $\left. x + y \leq 1 \right\}$. The minimal ideal K is $\left\{ \begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix} \middle| \, y \right.$ is a real number satisfying $\left. 0 \leq y \leq 1 \right\}$. Since K consists of left zeros, any subset of K is a semi-ideal of K; in particular, let $A = \left\{ \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0, & \frac{1}{2} \\ 0 & 1 \end{bmatrix} \right\}$. Since $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}$, then A is not a semi-ideal of S. Thus it follows from Proposition 3.1.15 that K is not agreeable with respect to S.

<u>3.3.6: Example.</u> This is an example of a semigroup T which is topologically a unit interval and such that the minimal ideal of T is not agreeable with respect to T.

Let S, K, and A be as in Example 3.3.5 and let $T = K \cup \left\{ \begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \middle| \, x \right.$ is a real number satisfying $\left. 0 \leq x \leq 1 \right\}$. The minimal ideal of T is K and A is a semi-ideal of K but not of T. It follows that K is not agreeable with respect to T.

<u>3.3.7: Example.</u> This example will serve to demonstrate that the converse of Proposition 3.1.15 is, in general, false.

Let T be as in Example 3.3.6, let $Y = [0,1]$ with the multiplication defined by $xy = y$, and let $S = T \times Y$ with co-ordinatewise multiplication. The minimal ideal K' of S is $K \times Y$ where K is the minimal ideal of T.

(3.3.7.1) If A is a semi-ideal of K, then card $A = 1$ or $A = K$.

Proof: Let A be a semi-ideal of K and suppose card $A > 1$. It follows that there are two distinct points, (x,y) and (u,w), in A and hence two cases to be considered: (1) $x \neq u$ and (2) $y \neq w$.

(1) $x \neq u$. If $z \in Y$, then, using the fact that K' consists of left zeros, it is seen that $(x,y)(x,z) = (x,z) \neq (u,z) = (u,w)(x,z) = (u,x)(x,z)$. Thus, since card $A(x,z) > 1$ and since A is a semi-ideal of K, then $(x,z) \in A^{[-1]}A$ so that $(u,z),(x,z) \in A$. This means $(\{x\} \times Y) \cup (u \times Y) \subset K$.

If $z \in K'$, $t \in Y$, and $t' \in Y$ such that $t' \neq t$, then $(x,t')(x,t) \in A$. But $(z,t)(x,t) = (z,t) \neq (z,t') = (z,t)(x,t')$ so that card $(z,t)A > 1$. It follows from the fact that A is a semi-ideal of K that $(z,t) \in AA^{[-1]}$. Thus, $(z,t) \in (z,t)A \subset A$ which means $K \subset A$.

(2) $y \neq w$. This argument is dual to that in case (1).

It is now clear that every semi-ideal of I is a semi-ideal of S. It remains only to show that K is not agreeable with respect to S.

Define the continuous function f' from K' onto K' by $f'\left(\begin{bmatrix} 0 & x \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 0 & 2x \\ 0 & 1 \end{bmatrix}$ if $x < \tfrac{1}{2}$ and $\begin{bmatrix} 0, & 1 \\ 0, & 1 \end{bmatrix}$ if $x \geq \tfrac{1}{2}$. Define the continuous function f'' from Y onto Y by $f''(x) =$ zx if $x \leq \tfrac{1}{2}$ and 1 if $x > \tfrac{1}{2}$. Define f from K to K by $f(x,y) = (f'(x),f''(y))$. The function f is clearly continuous and a simple computation shows it is a homomorphism.

It remains yet to show that (3.1) is not satisfied. Let $x = \begin{bmatrix} 0 & \tfrac{1}{2} \\ 0 & 1 \end{bmatrix}$, $y = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$, $z = \begin{bmatrix} \tfrac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$, and $t = \begin{bmatrix} 0 & \tfrac{1}{4} \\ 0 & 1 \end{bmatrix}$. It

follows that $f(x,0) = (y,0) = f(y,0) \neq f(t,0)$ but $(z,0)(y,0) = (z,0)$ and $(z,0)(x,0) = (t,0)$ so that (3.1) is not satisfied.

4: The Case in Which A Consists of Left Zeros.

Throughout this section, S is a semigroup and A is a closed subsemigroup of S. The agreeability of A with respect to S is investigated in the case where A consists of left zeros. It turns out that a condition weaker than A consisting of left zeros is needed for this investigation; this condition is called 2-simplicity. A theorem in this section gives necessary and sufficient conditions for a 2-simple closed subsemigroup to be agreeable.

3.4.1: Definition. A semigroup T is n-simple if and only if the cardinality of T is not less than n and every subset of T with n elements is a semi-ideal of T where $n > 0$. The semigroup is totally simple if and only if whenever n is not greater than the cardinality of T, then T is n-simple.

A few remarks concerning the structure of 2-simple semigroups are needed before the proof of the main result of this section. The following lemma will demonstrate that the idea of total simplicity is a generalization of the ideal of semigroups consisting of left or right zeros.

3.4.2: Lemma. If T is a semigroup such that $xT = x$ or $Tx = x$ for all $x \in T$, then T is totally simple.

Proof: Suppose $n \leq$ card T and let B be a subset of T with

cardinality n. If $x \in T$, then $xB = x(Bx = x)$ so that card $[xB \cap (T - B)] \leq 1$ (card$[Bx \cap (T - B)] \leq 1$). Since $Bx = B(xB = B)$, $x \in B^{[-1]}B$ ($x \in BB^{[-1]}$).

The type of homomorphisms which may be defined on a 2-simple semigroup is described in the next lemma.

<u>3.4.3: Lemma.</u> If P is a semigroup of cardinality of at least two, then P is 2-simple if and only if for every pair x,y of distinct points of P there exists a semigroup T and a continuous epimorphism f from P to T with the properties that $f(x) = f(y)$, f restricted to $P - \{x,y\}$ is a monomorphism, and $f[P - \{x,y\}] = T - \{f(x)\}$.

Proof: Suppose P is 2-simple and x,y are two distinct points of P. Since $\{x,y\}$ is a semi-ideal, it follows from Proposition 3.1.11 that $Q(\{x,y\},P)$ admits a desirable multiplication. If f is the natural projection from P onto $Q(\{x,y\},P)$, then f is easily seen to have the required properties.

Suppose $x,y \in P$ and let T and f be as described. Since P is agreeable with respect to P, $Z(f,P)$ admits a desirable multiplication. But $Z(f,P) = Q(\{x,y\},P)$, which according to Proposition 3.1.11 admits a desirable multiplication if and only if $\{x,y\}$ is a semi-ideal of P. Thus, P is 2-simple.

<u>3.4.4: Remark.</u> A group is 2-simple if and only if it is the cyclic group of order 2.

Theorem 3.4.9 gives a set of necessary and sufficient conditions for a 2-simple semigroup to be agreeable; this

theorem is the main result of this section. The main part of the proof to Theorem 3.4.9 is presented in Lemmas 3.4.5, 3.4.6, and 3.4.7.

3.4.5: Lemma. Let A be a subsemigroup of a semigroup S. If A is a semi-ideal of S and A satisfies conditions (1) and (2), then every semi-ideal of A is a semi-ideal of S.

(1) If $x \in AA^{[-1]}$, then card $xA = 1$ or $xa = a$ for all $a \in A$.

(2) If $x \in A^{[-1]}A$, then card $Ax = 1$ or $ax = a$ for all $a \in A$.

Proof: This follows from Proposition 3.2.5 and 3.1.16.

3.4.6: Lemma. Let A be a 2-simple subsemigroup of a semigroup S.

(i) If there are $x \in AA^{[-1]}$ and $a \in A$ such that $xa \neq a$ and card $xA > 2$, then there exists a semi-ideal of A which is not a semi-ideal of S.

(ii) If there is a point x in $A^{[-1]}A$ and a point a in A such that $ax \neq a$ and card $Ax > 2$, then there exists a semi-ideal of A which is not a semi-ideal of S.

Proof: (i) Since card $xA > 2$, there exists $a' \in xA$ such that $xa \neq a'$ and $a' \neq a$. From the fact that $a' \in xA$, it follows that there is $a'' \in A$ with the property $xa'' = a'$. Let $B = \{a,a''\}$; B is a semi-ideal of A because card $B = 2$.

Suppose B is a semi-ideal of S. Using the fact that $xa \neq a' = xa''$, it is seen that card $xB = 2$ so that $x \in BB^{[-1]}$. This implies that $xa, xa'' \in B$ and $xa = a''$ since $x \neq a$.

Also, it follows that $xa'' \neq a$, for if $xa'' = a$, then $a' = a$ and a' was chosen so that $a' \neq a$. Therefore, $xa'' = a''$ which implies $a'' = a'$ since $a' = xa''$. Thus, $xa = a'' = a'$, but this is a contradiction since a' was chosen in such a manner that $xa \neq a'$, and therefore B is not a semi-ideal of S.

The proof of (ii) follows in a similar fashion.

<u>3.4.7: Lemma.</u> Let A be a 2-simple subsemigroup of a semigroup S and let the cardinality of A be greater than two.

(i) If there exists a point x in $AA^{[-1]}$ such that card $xA = 2$, then there is a semi-ideal of A which is not a semi-ideal of S.

(ii) If there exists a point x in $A^{[-1]}A$ such that card $Ax = 2$, then there is a semi-ideal of A which is not a semi-ideal of S.

Proof: Since xA is a proper subset of A, there is a point a in $A - xA$; let $xa = a'$. There exists a point a' in $xA - \{a'\}$ since card $xA = 2$. Thus a, a', a'' are all distinct points and $xA = \{a', a''\}$.

There are three cases to be considered: (1) $xa' = a''$, (2) $xa' = a'$, $xa'' = a''$, and (3) $xa' = a'$, $xa'' = a'$.

(1) $xa' = a''$. If $B = \{a, a'\}$, then since A is 2-simple, B is a semi-ideal of A. If B is also a semi-ideal of S, then since $xa = a' \in B$, $x \in BB^{(-1)} \subset BB^{[-1]}$. Thus, $a'' = xa' \in B$ but $a' \notin B$, which is a contradiction. Therefore B is not a semi-ideal of S.

(2) $xa' = a'$ and $xa'' = a''$. If $B = \{a,a''\}$, then B is a semi-ideal since A is 2-simple. Since $xa'' = a''$, $x \in BB^{(-1)}$. But $xa = a' \in B$ so that $x \notin BB^{[-1]}$. Thus $BB^{(-1)}$ is not a subset of $BB^{[-1]}$ and B is not a semi-ideal of S.

(3) $xa' = a'$ and $xa'' = a'$. Since a'' is in xA, there is a point a''' in A such that $xa''' = a''$. Since $a' \neq a''$, $a''' \neq a$.

If $B = \{a,a'''\}$, then the 2-simplicity of A implies B is a semi-ideal of A. Since $xa = a'$ and $xa''' = a''$, it follows that $xB \subset (S - B)$. But card $xB = 2$ so that B is not a semi-ideal of S.

The proof of (ii) is similar to the proof of (i).

__3.4.8: Theorem.__ Let A be a 2-simple subsemigroup of a semigroup S such that the cardinality of A is larger than 2. Then every semi-ideal of A is a semi-ideal of S if and only if the conditions (1), (2), and (3) are satisfied.

(1) If $x \in AA^{[-1]}$, then card $xA = 1$ or $xa = a$ for all a in A.

(2) If $x \in A^{[-1]}A$, then card $Ax = 1$ or $ax = a$ for all a in A.

(3) A is a semi-ideal of S.

Proof: This theorem follows from Lemmas 3.4.5, 3.4.6, and 3.4.7.

__3.4.9: Theorem.__ Let A be a 2-simple compact subsemigroup of a compact semigroup S such that the cardinality of A is greater than two. Then A is agreeable with respect to S if

and only if every semi-ideal of A is a semi-ideal of S.

Proof: Necessity. This follows immediately from Proposition 3.1.15.

Sufficiency. Theorem 3.4.8 says that the hypotheses of Proposition 3.2.5 are satisfied.

<u>3.4.10: Corollary.</u> Let A be a compact 2-simple subsemigroup of a compact semigroup S. If the cardinality of A is greater than two, then A is agreeable with respect to S if and only if the conditions (1), (2), and (3) are satisfied.

(1) If $x \in AA^{[-1]}$, then card $xA = 1$ or $xa = a$ for all a in A.

(2) If $x \in A^{[-1]}A$, then card $Ax = 1$ or $ax = a$ for all a in A.

(3) A is a semi-ideal of S.

<u>3.4.11: Corollary.</u> A semigroup T with cardinality greater than two is 2-simple if and only if the following two conditions are satisfied.

(1) If $x \in TT^{[-1]}$, then card $xT = 1$ or $xa = a$ for all a in T.

(2) If $x \in T^{[-1]}T$, then card $Tx = 1$ or $ax = a$ for all a in T.

Proof: The necessity is immediate from Theorem 3.4.8 with $T = S = A$.

Sufficiency. Suppose a and a' are distinct points of T, let $x \in T$ and let $B = \{a,a'\}$. If card $xA = 1$, then card $xB = 1$. If card $xA = 1$, then $xB = B$ so that $x \in BB^{[-1]}$.

The argument for right multiplication is similar.

3.4.12: Corollary. Let A be a closed subsemigroup of a compact semigroup S. If the cardinality of A is larger than two and if A consists of left zeros of S, then A is agreeable with respect to S if and only if $x \in AA^{[-1]}$ implies card $xA = 1$ or $xa = a$ for all a in A.

Proof: Using Lemma 3.4.2, it is seen that A is 2-simple, and so Corollary 3.4.10 is applicable. Since A consists of left zeros of S, (2) of 3.4.10 is satisfied.

For right zeros, a statement similar to 3.4.12 is made.

3.4.13: Corollary. Let A be a compact subsemigroup of a compact semigroup S. If the cardinality of A is greater than two and if A consists of right zeros of S, then A is agreeable with respect to S if and only if $x \in A^{[-1]}A$ implies card $xA = 1$ or $ax = a$ for all a in A.

5: Agreeable Minimal Ideals.

In this section, S is a compact semigroup and K is the minimal ideal of S. A theorem which gives necessary and sufficient conditions for K to be agreeable with respect to S will be proved in this section. Various structure theorems for K will be used in this section. These theorems may be found in [2], [10], [16], [18], and [20].

3.5.1: Lemma. Let A be a compact subsemigroup of S, let f be a continuous epimorphism from A onto P, let φ be the natural projection of S onto $Z(f,S)$, and let k be the

natural embedding of P into Z(f,S). If A is agreeable with respect to S, then k(P) is agreeable with respect to Z(f,S). Moreover, if g is a continuous epimorphism from P onto T, then $g' = gk^{-1}$, and if φ' and φ'' are the natural projections of Z(f,S) and S onto Z(g', Z(f,S)) and Z(gf,S), respectively, then there exists an iseomorphism h from Z(gf,S) into Z(g',Z(f,S))

$$\begin{tikzcd} S \arrow[r, "\varphi"] & Z(f,S) & S \arrow[r, "\varphi''"] & Z(gf,S) \\ A \arrow[u, phantom, "\cup"] \arrow[r, "f"] & P \arrow[u, "k"'] & A \arrow[u, phantom, "\cup"] \arrow[r, "gf"] & T \arrow[u, "k''"'] \end{tikzcd}$$

$$\begin{tikzcd} S \arrow[r, "\varphi"] & Z(f,S) \arrow[r, "\varphi'"] & Z(g',Z(f,S)) \\ A \arrow[u, phantom, "\cup"] \arrow[r, "kf"] & k(P) \arrow[u, phantom, "\cup"] \arrow[r, "g'"] & T \arrow[u, "k'"'] \end{tikzcd}$$

Proof: Since semi-ideals and semigroups are preserved under epimorphisms, $k(P) = \varphi(P)$ is a semigroup and semi-ideal. Thus it is enough to verify (3.1) for a continuous epimorphism g' from k(P) onto T. Suppose $\varphi(x) \in [k(P)]^{[-1]}[k(P)]$ and $g'\varphi(a) = g'\varphi(a')$. Since A is agreeable with respect to S, (3.1) must hold for the epimorphism $g'\varphi$. Thus $g'[\varphi(x)\varphi(a)] = g'\varphi(xa) = g'\varphi(xa') = g'[\varphi(x)\varphi(a)]$ and k(P) is agreeable with respect to Z(f,S).

Let h be defined by $h(z) = \varphi'\varphi\varphi''^{-1}(z)$. It has been shown previously that h is a homeomorphism, so that it is enough to verify that h is a homomorphism. Using the homomorphic properties of φ, φ', and φ'', it is seen that $h[\varphi''(x)\varphi''(y)] = h\varphi''(xy) = \varphi'\varphi(xy) = \varphi'\varphi(x)\varphi'\varphi(y) = h(\varphi''(x))h(\varphi''(y))$.

Notation: The function u from K into E is defined by $u(x) = e$ if and only if $x \in eS \cap Se$. Wallace [20] has shown that u is continuous. Through Proposition 3.5.4 e will be a fixed point in $K \cap E$, P will denote $Se \cap E$, and f will be the function from K to P defined by $f(x) = u(xe)$. It should be noted that P is a subsemigroup which is 2-simple, since $pq = p$ for all $p,q \in P$.

<u>3.5.2: Lemma.</u> The function f is a homomorphic retraction of K onto P.

Proof: Clearly, f is continuous since multiplication and u are. If $p \in Se \cap E$, then $f(p) = p$ since $pe = p$ and $u(p) = p$. It remains only to show that f is a homomorphism. If $x,y \in K$, then $xye \in xS \cap Se = xeS \cap Se = f(x)S \cap Sf(y)$ so that $f(xy) = u(xye) = f(x) = f(x)f(y)$.

<u>3.5.3: Lemma.</u> If $x,y \in K$, then $f(x) = f(y)$ if and only if $xS = yS$.

Proof: If $f(x) = f(y)$, then $f(x)S = f(y)S$ and $xS = xeS = f(x)S = f(y)S$

If $xS = yS$, then, using the fact that $wS \cap Sp$ is a subgroup of S for all $w,p \in K$, it is seen that $f(x) = xeS \cap Se \cap E = xS \cap Se \cap E = yS \cap Se \cap E = yeS \cap Se \cap E = f(y)$.

<u>3.5.4: Proposition.</u> If K is agreeable with respect to S and if there are at least three minimal right ideals of S, then K satisfies 5.31.

5.31: If $x \in S$, then $xt = tS$ for all t in K or there exists a point t' in K such that $xK = t'S$.

Proof: In the following analytic diagram k is an isomorphism.

$$\begin{array}{ccc} S & \xrightarrow{\varphi} & Z(f,S) \\ \cup & & \uparrow k \\ K & \xrightarrow{f} & P \end{array}$$

Since k(P) is agreeable with respect to S, it follows from Lemma 3.5.1 that k(P) is agreeable with respect to Z(f,S). Corollary 3.4.10 is applicable since k(P) is 2-simple. Thus $k(P)(k(P)^{[-1]}) = Z(f,S)$ since k(P) is an ideal of Z(f,S). Therefore, it follows from Corollary 3.4.10 that if $x \in S$, then $\varphi(x)\varphi(t) = \varphi(t)$ for all t in K or there exists a point t' in K such that $\varphi(x)\varphi(t) = \varphi(t')$ for all t in K. If $\varphi(x)\varphi(t) = \varphi(t)$, then $\varphi(xt) = \varphi(t)$ so that $xtS = tS$. If $\varphi(x)\varphi(t) = \varphi(t')$ for all $t \in K$, where t' is a fixed point in K, then $\varphi(xt) = \varphi(t')$ so that $xtS = t'S$. This means that $xK = x(\cup\{tS \mid t \in K\}) = t'S$.

The following proposition may be proved in a manner dual to the proof of the previous proposition.

<u>3.5.5: Proposition.</u> If K is agreeable with respect to S and if there are at least three minimal left ideals of S, then K satisfies 5.32.

5.32: If $x \in S$, then $Stx = St$ for all t in K or there is a point t' in K such that $Kx = St'$.

Next, it is shown that if K satisfies 5.31 and 5.32, then K is agreeable with respect to S. To demonstrate this fact, we shall verify that an arbitrary epimorphism f

defined on K satisfied 3.1.9; this verification is done in the next few lemmas.

If $e \in E$, then there exists a subgroup H_e which is maximal with respect to containing e [10]. If $e \in E \cap K$, then $H_e = eS \cap Se$ [10]. These two theorems are used extensively in the remainder of this section.

3.5.6: Lemma. If $f(k) = f(k')$, then $f(u(k)) = f(u(k'))$.

Proof: Since $f[H_{u(k)}]$ and $f[H_{u(k')}]$ are subgroups of $f[K]$, there exists $y,y' \in T$ such that $f(k)y = yf(k) = f(u(k))$ and $f(k')y' = y'f(k') = f(u(k'))$. Thus, using the fact that $u(k)$ and $u(k')$ act as units on $H_{u(k)}$ and $H_{u(k')}$, respectively, it is seen that $f(u(k)) = f(k)y = f(k')y = f(u(k'))$ $f(k')y = f(u(k'))f(k)f(y) = y'f(k')f(u(k)) = y'f(k)f(u(k))$ $= y'f(k) = y'f(k') = f(u(k'))$.

3.5.7: Lemma. Let $k,k' \in K$ such that $f(k) = f(k')$ and let $x \in S$.

(i) If $f(u(xu(k))) = f(u(xu(k')))$, then $f(xk) = f(xk')$.

(ii) If $f(u(u(k)x)) = f(u(u(k)x))$, then $f(kx) = f(h'x)$.

Proof: (i) Let $e' = u(k)$, $e'' = u(k')$, $e_1 = u(xe')$, and $e_2 = u(xe'')$. Using Lemma 3.5.6, it is seen that $f(xe') = f(e_1xe') = f(e_1)f(xe) = f(e_2)f(xe') = f(e_2x)$ $f(e') = f(e_2x)f(e'') = f(e_2xe'') = f(xe'')$ so that $f(xk) =$ $f(xk) = f(xe'k) = f(xe')f(k) = f(xe'')f(k') = f(xe''k') =$ $f(xk')$.

The proof of (ii) follows by an argument similar to the above argument.

<u>3.5.8: Lemma.</u> Let e', e'', e_1, e_2 ε $K \cap E$ such that $f(e') = f(e'')$.

(i) If $e_1S = e_2S$, $Se' = Se_1$, and $Se'' = Se_2$, then $f(e_1) = f(e_2)$.

(ii) If $Se_1 = Se_2$, $e'S = e_1S$, and $Se'' = Se_2$, then $f(e_1) = f(e_2)$.

Proof: (i) Since e_2, e'' ε $Se'' \cap E$, e_2e'' ε $Se'' \cap E \cap e_2S = Se_2 \cap E \cap e_2S = \{e_2\}$. Using the homomorphic properties of f and the hypotheses, it is seen that $f(e_2) = f(e_2e'') = f(e_2)f(e'') = f(e_2)f(e') = f(e_2e')$ ε $f(e_2S \cap Se') = f(e_1S \cap Se_1) = f[He_1]$. Therefore $f(e_1) = f(e_2)$, since $f(e_2)$ is an idempotent and $f[He_1]$ is a group.

The proof of (ii) follows from a similar argument.

<u>3.5.9: Lemma.</u> Let p,k ε K and let x ε S.

(i) If $Skx = Sp$, then $Su(k)x = Sp$.

(ii) If $xkS = pS$, then $xu(k)S = pS$.

The proof of this lemma is obvious.

<u>3.5.10: Lemma.</u> Let k,k' ε K such that $f(k) = f(k')$ and let x ε S.

(i) If $xkS = kS$ and $xk'S = k'S$, then $f(xk') = f(xk)$.

(ii) If $Skx = Sk$ and $Sk'x = Sk'$, then $f(k'x) = f(kx)$.

Proof: (i) Let $e' = u(k)$, $e'' = u(k')$, $e_1 = u(xe')$, and $e_2 = u(xe'')$. It follows that $e_1 = u(xe') = e'$, since $xe' \in Se' \cap xe'S = Se' \cap xkS = Se' \cap kS = Se' \cap e'S$. Similarly, $e_2 = e''$. Thus, from Lemma 3.5.6, $f(e_1) = f(e_2)$ and hence $f(xk) = f(xk')$ by Lemma 3.5.7.

The proof of (ii) is similar.

<u>3.5.11: Lemma.</u> Let $k,k' \in K$ such that $f(k) = f(k')$ and let $x \in S$.

(i) If $xkS = xk'S = t'S$ for some t' in K, then $f(xk) = f(xk')$.

(ii) If $Skx = Sk'x = St''$ for some t'' in K, then $f(k'x) = f(kx)$.

Proof: (i) Let $e' = u(k)$, $e'' = u(k')$, $e_1 = u(xe')$, and $e_2 = u(xe'')$. It follows that $Se_1 = Se'$ and $e_1S = t'S$ since $xe' \in Se' \cap xe'S = Se' \cap t'S$. Similarly, $Se_2 = Se''$ and $e_2S = t'S$. Therefore, from Lemma 3.5.6, $f(e') = f(e'')$ and since $e_1S = u(xe')S = xe'S = xkS = t'S = xk'S = xe''S = u(xe'')S = e_2S$ and $e_1S = t'S = e_2S$, the hypotheses of Lemma 3.5.8 are satisfied. Thus $f(e_1) = f(e_2)$, and hence, from Lemma 3.5.7, $f(xk) = f(xk')$.

The proof of (ii) follows by a similar argument.

<u>3.5.12: Proposition.</u> If K satisfies 5.31 and 5.32, then K is agreeable with respect to S.

Proof: This proposition follows immediately from Lemmas 3.5.10 and 3.5.11.

3.5.13: Lemma.

(i) If S contains two minimal right ideals and if $f(k) = f(k')$, then $f(xk) = f(xk')$ for all $x \in S$.

(ii) If S contains two minimal left ideals and if $f(k) = f(k')$, then $f(kx) = f(k'x)$ for $x \in S$.

Proof: (i) There are four cases to be considered: (1) $xkS = kS$, $xk'S = k'S$; (2) $xkS = ks$, $xk'S = ks$; (3) $xkS = k'S$, $xk'S = k'S$; and (4) $xkS = k'S$, $xk'S = kS$.

(1) follows from Lemma 3.5.10 (i).

(2) and (3) follow from Lemma 3.5.11 (ii).

(4) Let $e' = u(k)$, $e'' = u(k')$, $g = u(xe')$, and $e_2 = u(xe'')$. It follows that $g = u(e''e')$ since $xe' \in Se' \cap xe'S = Se' \cap k'S = Se' \cap e''S$.

Using Lemma 3.5.6 and the idempotentcy property of e'', it is seen that $f(e'') = f(e'')f(e'') = f(e'')f(e') = f(e''e') \in f[H_g]$. But $f(e'')$ is also an idempotent so that $f(e'') = f(g)$. Similarly, $f(e') = f(e_2)$ so that $f(g) = f(e'') = f(e') = f(e_2)$. Thus, the conclusion follows from Lemma 3.5.7 and (i) holds in all four cases.

The proof of (ii) is similar.

3.5.14: Proposition.

(i) If S contains two minimal right ideals and two minimal left ideals, then K is agreeable with respect to S.

(ii) If S contains two minimal right ideals and if S and K satisfy 5.32, then K is agreeable

with respect to S.

(iii) If S contains two minimal left ideals and if K satisfies 5.31, then K is agreeable with respect to S.

Proof:

(i) follows from 3.5.13.

(ii) follows from 3.5.13 (i), 3.5.10 (ii), and 3.5.11 (ii).

(iii) follows from 3.5.13 (ii), 3.5.10 (i), and 3.5.11 (i).

The following theorem combines Proposition 3.5.5, 3.5.6, 3.5.12, and 3.5.14.

3.5.15: Theorem. If K is the minimal ideal of a compact semigroup S, then K is agreeable with respect to S if and only if at least one of the following statements hold:

(1) 5.31 and 5.32 are satisfied by K and S.

(2) There are exactly two minimal left ideals of S and 5.31 is satisfied by S and K.

(3) There are exactly two minimal right ideals of S and 5.32 is satisfied by S and K.

(4) There are exactly two minimal left ideals and exactly two minimal right ideals of S.

6. Congruences.

Let S be a compact semigroup, let A be a closed subsemigroup of S, let $\mathfrak{C}$ be the set of all closed congruences of A, and let $\mathfrak{H}$ be the set of all epimorphisms defined on A.

Recall that there exists a one-to-one correspondence between $\mathfrak{C}$ and $\mathfrak{J}$. In the previous two sections a subset $\mathfrak{G}$ of $\mathfrak{J}$ was chosen in order to prove the necessity of the two theorems in those sections. In a certain sense, each element of $\mathfrak{G}$ was minimal in $\mathfrak{J}$. The fact that it is enough to look at $\mathfrak{G}$ rather than $\mathfrak{J}$ is not accidental. It is the purpose of this section to explain why this is true. However, it is simpler to look at $\mathfrak{C}$ rather than $\mathfrak{J}$ for this purpose.

Because of the natural correspondence between $\mathfrak{J}$ and $\mathfrak{C}$, the results of this section can be readily interpreted for $\mathfrak{J}$.

First, a few preliminary remarks concerning congruences need to be made.

3.6.1: Lemma. Let $\mathfrak{A}$ be a collection of closed congruences of A. Then $B' = \cap \{B \mid B \in \mathfrak{A}\}$ is a closed congruence of A.

Proof: It is clear that B' is a closed equivalence relation on A. It remains to show that $(\Delta A)\, B' \subset B' \supset B'(\Delta A)$. If $x \in A$ and $(b,b') \in B$ for all $B \in \mathfrak{A}$, then $(xb,xb') \in B$ for all $B \in \mathfrak{A}$ and $(\Delta A)\, B' \subset B'$. Similarly, B' contains $B'(\Delta A)$.

3.6.2: Lemma. If $x,y \in A$, then there exists a unique minimal closed congruence of A containing (x,y). This congruence will be denoted by $F(x,y)$.

Proof: Let $F(x,y)$ be the intersection of all closed congruences containing (x,y).

3.6.3: Theorem. If A is a closed subsemigroup of the compact semigroup S, then A is agreeable with respect to S if

and only if $F(a,a') \cup (\Delta S)$ is a congruence of S for all $a,a' \in A$.

Proof: The necessity follows immediately from Remark 3.1.14.

Sufficiency. Suppose B is a congruence of A. It is sufficient, according to Remark 3.1.14, to show that $B \cup (\Delta S)$ is a congruence of S, i.e., if $x \in S$ and $(w,z) \in B \cup (\Delta S)$, then (xw,xz), $(wx, zx) \in B \cup (\Delta S)$. This is clear if $(w,z) \in \Delta S$. If $(w,z) \in B$, then (xw,xz), $(wx,zx) \in F(w,z) \cup (\Delta S)$ since $F(w,z) \cup (\Delta S)$ is a congruence of S. It follows that (xw,xz), $(wx,zx) \in B \cup (\Delta S)$ since $F(w,z)$ is a subset of B.

CHAPTER IV

HOMOMORPHIC RETRACTS IN SEMIGROUPS

In the previous chapter the concept of homomorphic retraction was introduced to study Borsuk's Paste Job in semigroups. The question of when is a subsemigroup a homomorphic retract arises because of the role that homomorphic retractions play in the hypothesis of Proposition 3.2.2. This question is studied in the present chapter. A set of necessary and sufficient conditions for the minimal ideal to be a homomorphic retract is given in Section 2 and it is shown how these conditions are applicable to subsemigroups other than the minimal ideal. A theorem which gives another set of necessary and sufficient conditions for the minimal ideal to be a homomorphic retract is presented in Section 3.

1: Preliminaries and Homomorphic Extensions.

Notation: In this section, S is a semigroup and A is a subsemigroup of S.

If A is agreeable with respect to S, then Z(f,S) is a "natural" semigroup for all continuous epimorphisms f which are defined on A. If f(A) is identified with its iseomorphic image in Z(f,S), then the natural projection

of S onto $Z(f,S)$ may be thought of as a continuous homomorphic extension of f. Thus, if the special features of $Z(f,S)$ are disregarded except for the fact that $f(A)$ can be iseomorphically embedded in $Z(f,S)$, then the previous chapter can be regarded as an attempt to construct a semigroup Z containing $f(A)$ as a subsemigroup such that f has a continuous homomorphic extension to S. In this chapter, the additional restriction that $Z = f(A)$ is made. The following theorem shows that this problem is equivalent to studying the problem of determining when a subsemigroup is a homomorphic retract.

<u>4.1.1: Theorem.</u> Every continuous epimorphism defined on A has a homomorphic extension to S if and only if A is a homomorphic retract of S.

Proof: Assume every epimorphism defined on A has a homomorphic extension to S. In particular, the identity function from A onto A has a homomorphic extension r to S, and this function r is a homomorphic retraction of S onto A.

Assume A is a homomorphic retract of S and let f be a continuous epimorphism defined on A. If r denotes a homomorphic retraction of S onto A, then rf is a homomorphic extension of f to S.

Along this same line, it is easily shown that every continuous homomorphism defined on A has a continuous (not necessarily homomorphic) extension to S if and only if A is a retract of S. Thus every continuous function defined on A has a continuous extension to S if and only if every

continuous homomorphism defined on A has a continuous extension to S. The following remark is similar.

4.1.2: Remark. The following are equivalent:

(1) There exists a continuous function f from S onto A such that the restriction of f to A is a homomorphism.*

(2) There exists a continuous epimorphism h from A onto A with the property that if g is a continuous homomorphism defined on A, then there exists a continuous function k from S to g(A) such that the following diagram is analytic.

$$\begin{tikzcd} S \arrow[r, "k"] & g(A) \\ \cup & \\ A \arrow[r, "h"] & A \arrow[uu, "g"'] \end{tikzcd}$$

Proof: (1) implies (2). Let h be the restriction of f to A. If g is a continuous homomorphism defined on A, then k = gf has the required properties.

(2) implies (1). If g is the identity from A onto A, then there exists a continuous function f such that the following diagram is analytic.

$$\begin{tikzcd} S \arrow[r, "f"] & g(A) \\ \cup & \\ A \arrow[r, "h"] & A \arrow[uu, "g"'] \end{tikzcd}$$

*This condition was suggested by Professor W. L. Strother.

Thus f restricted to A is h since $f(S) = A$ and f is continuous.

We now turn our attention to the problem of determining when A is a homomorphic retract of S; the following example will demonstrate that the minimal ideal of a compact semigroup may be a homomorphic retract of the semigroup without the minimal ideal being a group.

4.1.3: Example. Let G be a non-trivial group and let $X = Y = Z = [0,1]$. Define $xy = x$ for all $x,y \in X$, and define $xy = y$ for all $x,y \in Y$. Let Z have the usual multiplication. Let $S = X \times Y \times Z \times G$ with co-ordinatewise multiplication. Clearly, the minimal ideal is $X \times Y \times \{0\} \times G$. Define r from S to K by $r(x,y,z,g) = (x,y,0,g)$. Since $r[(x,y,z,g)(x',y',z',g')] = r(x,y',zz',gg') = (x,y',0,gg')$ $= (x,y,0,g)(x',y',0,g') = r(x,y,z,g)r(x',y',x',g')$, r is a homomorphic retraction.

The remainder of this section contains two propositions which will be used later.

4.1.4: Proposition. A retraction f of a semigroup S onto a subset B is a homomorphic retraction if and only if B is a subsemigroup of S and $f(xy) = f(f(x)f(y))$ for all $x,y \in S$.

Proof: The necessity of this condition is obvious.

If B is a subsemigroup of S and if $f(xy) = f(f(x)f(y))$ for all $x,y \in S$, then $f(xy) = f(x)f(y)$ since $f(x)f(y) \in B$. Thus f is a homomorphic retraction.

4.1.5: Proposition. Let f be a homomorphic retraction of

S onto T and let R be a congruence of S such that $f(xR) = f(x)R$ for all $x \in S$. Then there exists a homomorphic retraction f^* of S/R onto $\varphi(T)$ where φ is the natural projection of S onto S/R. Moreover, the following diagram is analytic.

$$\begin{tikzcd} S/R \arrow[r, "f^*"] & S/R \\ S \arrow[u, "\varphi"] \arrow[r, "f"] & S \arrow[u, "\varphi"'] \end{tikzcd}$$

Proof: The fact that there exists a continuous homomorphism f^* which makes the above diagram analytic is well known. Therefore, it is sufficient to show that f^* is a retraction of S/R onto $\varphi(T)$.

If $x \in S$, then $f^*\varphi(x) = \varphi f(x) \in \varphi(T)$. If $z \in \varphi(T)$, then there exists $x \in T$ such that $\varphi(x) = z$. Thus $f^*(z) = f^*\varphi(x) = \varphi f(x) = \varphi(x) = z$ so that f^* is a homomorphic retraction of S/R onto $\varphi(T)$.

2. The First Theorem.

In Sections 2 and 3 it is necessary to make use of various structure theorems for the minimal ideal of a semigroup. These may be found in [2], [10], [16], [18], and [20]. In the remainder of this chapter, K is the minimal ideal of S.

4.2.1: Proposition. Let R be a right ideal of a semigroup S and let there exist a continuous function f from S into R such that f has the following properties:

(1) $f(xy)xy = f(x)x\, f(y)y$ for all $x,y \in S$,

(2) $f(x)x = x$ for all $x \in R$;

then R is a homomorphic retract of S.

Proof: The function r from S to R is defined by $r(x) = f(x)x$ for all $x \in S$. It will be shown that the function r is a homomorphic retraction of S onto R.

Since multiplication and f are continuous, r is continuous. It follows that r is a retraction because (2) implies $r(x) = x$ for all $x \in R$. Also, by (1) it is seen that $r(xy) = f(x)xf(y)y = r(x)r(y)$ for all $x,y \in S$ so that r is a homomorphism.

<u>4.2.2: Proposition.</u> Let L be a left ideal of a semigroup S and let there exist a continuous function f from S into L such that f has the following properties:

(1) $xyf(xy) = xf(x)yf(y)$ for all $x,y \in S$,

(2) $xf(x) = x$ for all $x \in L$;

then L is a homomorphic retract of S.

The proof of this proposition is dual to the proof of Proposition 4.2.1.

<u>4.2.3: Corollary.</u> Let R be a right ideal of a semigroup S. If R contains a point which acts as a unit for R, then R is a homomorphic retract of S.

Proof: Let $a \in R$ such that a is a unit for R. Define f from S into R by $f(x) = a$ for all $x \in S$. To establish the corollary it is sufficient to verify that f satisfies the hypotheses of Proposition 4.2.1. Clearly, f is continuous, and using the fact that a is a right unit for the points

of R, $f(xy)xy = axy = (ax)y = (axa)y = (ax)(ay) = (f(x)x)(f(y)y)$ for all $xy \in S$. Also, if $x \in R$, then the fact that a is a left unit for the points of R implies that $f(x)x = ax = x$.

<u>4.2.4: Corollary.</u> Let L be a left ideal of a semigroup S. If L contains a point which acts as a unit for L. then L is a homomorphic retract of S.

The proof of this corollary is dual to the proof of Corollary 4.2.3.

The next corollary appears in [8, p. 289].

<u>4.2.5: Corollary.</u> Let I be an ideal of a semigroup S. If I has an identity for I, then I is a homomorphic retract of S.

The proof of Corollary 4.2.5 is immediate from Corollary 4.2.4.

Notation: In the rest of this chapter S is a compact semigroup, K is the minimal ideal of S, E denotes the set of idempotents of S, and $H = \cup\{H_e | e \in E\}$ where H_e is the maximal subgroup of S which contains e.

<u>4.2.6: Lemma.</u> Let A be a closed subset of S and let f be a function from S to A with the following properties:

(1) $f(x)x = xf(x)$ for all x in S,

(2) if $y \in A$ and $yx = xy$ for some x in S, then $y = f(x)$.

Then f is continuous.

Proof:* Let m be the function defined from $S \times S$ to $S \times S$

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*This proof was suggested by Professor A. D. Wallace.

by $m(x,y) = (xy,yx)$. m is clearly continuous so that $m^{-1}[\Delta S \cap (S \times A)]$ is closed. But $m^{-1}[\Delta S \cap (S \times A)] = \{(x,y): xy = yx \text{ and } y \in A\} = \{(x,f(x)): x \in S\}$ so that f is continuous.

<u>4.2.7: Definition.</u> A function f from S into S is a <u>reduction</u> of S if and only if f has the following properties:

4.2.7.1 If $x \in S$ and if $y \in f(S)$, then $f(x) = y$ if and only if $xy = yx$.

4.2.7.2 If $x,y \in S$, then $f(xy)xyf(xy) = f(x)xyf(y)$.

A subset A of S is called a <u>reduct</u> of S if and only if there exists a reduction f of S such that $f(S) = A$.

<u>4.2.8: Proposition.</u> If the image of a reduction is closed, then the reduction is a retraction.

Proof: If f is a reduction, then Lemma 4.2.4 and 4.2.7.1 imply f is continuous and the fact $f(a)$ commutes with $f(a)$ for all $a \in S$ implies that $f^2 = f$; thus f is a retraction.

<u>4.2.9: Proposition.</u> Let R be a closed right ideal of S contained in $\cup\{eSe \mid e \in R \cap E\}$. If $R \cap E$ is a reduct of S, then R is a homomorphic retract of S.

Proof: Let f denote the reduction of S onto $R \cap E$. It is sufficient to verify that f satisfies the hypotheses of Proposition 4.2.1. Lemma 4.2.4 guarantees that f is continuous. Using 4.2.7.1 and 4.2.7.2, it is seen that $f(xy)xy = f(xy)f(xy)xy = f(xy)xyf(xy) = f(x)xyf(y) = f(x)xf(y)y$ for all $x,y \in S$. If x is a point in R, then there exists a point e in $R \cap E$ such that $x \in eSe$. Thus

$ex = x = xe$ so that $f(x) = e$.

<u>4.2.10: Corollary.</u> Let R be a closed right ideal of S contained in $H = \cup \{H_e | e \in E\}$. If $R \cap E$ is a reduct of S, then R is a homomorphic retract of S.

Proof: If $x \in R$, then there exists $e \in E$ such that $x \in H_e$. Since H_e is a group, there exists a point $y \in H_e$ such that $xy = e$ so that $e \in R$. Thus R is contained in $\cup \{eSe | e \in E \cap R\}$ and it follows from Proposition 4.2.9 that R is a homomorphic retract of S.

The following two statements are the duals of the previous two statements.

<u>4.2.11: Proposition.</u> Let L be a closed left ideal of S contained in $\cup \{ese | e \in L \cap E\}$. If $L \cap E$ is a reduct of S, then L is a homomorphic retract of S.

<u>4.2.12: Corollary.</u> Let L be a closed left ideal of S contained in H. If $L \cap E$ is a reduct of S, then L is a homomorphic retract of S.

<u>4.2.13: Proposition.</u> Let I be an ideal of S which is contained in H. If I is a homomorphic retract of S, then there exists a continuous function f from S into $E \cap I$ with the properties that $f(x)x = xf(x)$ and that $f(xy)xyf(xy) = f(x)xyf(y)$ for all x and y in S.

Proof: Let r be a homomorphic retract of S onto I and define f from S into $E \cap I$ by $f = ur$. Since r and u are continuous, f is continuous. Using the definitions of u and f and the homomorphic retraction properties of r, it is seen that $xf(x) = r(xf(x)) = r(x)rf(x) = r(x)f(x) = r(x) =$

$f(x)r(x) = rf(x)r(x) = r(f(x)x) = f(x)x$ for all x in S. Also, $f(xy)xyf(xy) = r(f(xy)xy)f(xy) = rf(xy)r(xy)f(xy) = f(xy)r(xy)f(xy) = r(xy) = r(x)r(y) = f(x)r(x)r(y)f(y) = rf(x)r(x)r(y)rf(y) = r(f(x)xyf(y)) = f(x)xyf(y)$ for all $x,y \in S$.

<u>4.2.14: Example.</u> This is an example of a semigroup S with an ideal I which is contained in H and which is a homomorphic retract of S, but $I \cap E$ is not a reduct of S.

Let S be the unit interval with the multiplication given by $xy = \min(x,y)$ and let $I = [0,\frac{1}{2}]$. Since $I \cap E = I$ and S is Abelian, no function from S onto $I \cap E$ will satisfy 4.2.7.1.

<u>4.2.15: Theorem.</u> The minimal ideal K is a homomorphic retract of S if and only if $K \cap E$ is a reduct of S.

Proof: The necessity follows from Corollary 4.2.7.

Define f from S to $K \cap E$ by $f = ur$ where r is a homomorphic retraction of S onto K. It follows from Proposition 4.2.9 that f satisfies 4.2.5.1, 4.2.5.3, and 4.2.5.4. It remains only to verify 4.2.5.2.

Suppose g is a function from S onto $K \cap E$ such that $g(x)x = xg(x)$ for all x in S. Let x be a fixed point in S. It must be shown $g(x) = f(x)$. Since $f = ur$, $r(x)\cdot vr(x) = f(x) = vr(x)\cdot r(x)$. Thus $g(x)f(x) = g(x)r(x)vr(x) = r(g(x)x)vr(x) = r(xg(x))vr(x) = r(x)g(x)vr(x) \in f(x)S \cap g(x)S$ so that $f(x)S = g(x)S$. In a similar fashion it may be shown that $Sf(x) = Sg(x)$. Therefore, $\{g(x)\} = g(x)S \cap Sg(x) \cap E = f(x)S \cap Sf(x) \cap E = \{f(x)\}$.

It is instructive to look at the case when S is simple, i.e., $S = K$. Clearly, K is a homomorphic retract of S so that there exists a reduction of S onto E. This reduction is u and it is clear that $u(x) = x = xu(x)$ for all $x \in S$. If $e \in E$ and $ex = xe$, then $ex \in eS \cap u(x)S \cap Su(x) \cap Se$. Thus $\{e\} = eS \cap E \cap Se = u(x)S \cap E \cap Su(x) = \{u(x)\}$. The next corollary says that if in an arbitrary semigroup S there exists a reduction of S onto $K \cap E$, then this reduction is an extension of u.

<u>4.2.16: Corollary.</u> If I is a closed ideal contained in H and if f is a reduction of S onto $I \cap E$, then I is a homomorphic retract of S, and if r is a homomorphic retraction of S onto I, then $f = ur$. Thus, there is only one homomorphic retraction of S onto I.

Proof: The fact that I is a homomorphic retract of S is exactly the statement of Proposition 4.2.9.

If $x \in S$, then, using the homomorphic retraction properties of r, it is seen that $xur(x) = r(xur(x)) = r(x)rur(x) = r(x)ur(x) = r(x) = ur(x)r(x) = rur(x)r(x) = r(ur(x)x) = ur(x)x$. Thus it follows from 4.2.7.2 that $f(x) = ur(x)$ for all $x \in S$.

It remains to show that there is only one homomorphic retraction of S onto I. This is done by showing that if r is a homomorphic retraction of S onto I, then $r(x) = f(x)x$. Using the definition of u and the fact that $f = ur$, it is seen that $r(x) = ur(x)r(x) = f(x)r(x) = rf(x)r(x) = r(f(x)x) = f(x)x$.

<u>4.2.17: Corollary.</u> If f is a reduction of S onto $K \cap E$, then $f(xy) = f(f(x)f(y))$ for all $x,y \in S$.

Proof: If $x,y \in S$, then $f(xy)xyf(xy) = f(x)xyf(y) \in f(xy)Sf(xy) \cap f(x)Sf(y)$. Thus $H_{u(f(x)f(y))} = H_{f(xy)}$ so that $u(f(x)f(y)) = f(xy)$. But $u(f(x)f(y)) = ur(f(x)f(y)) = f(f(x)f(y))$ where r is any homomorphic retraction of S onto K.

<u>4.2.18: Corollary.</u> If K is a homomorphic retract of S and if $K \cap E$ is a subsemigroup of S, then $K \cap E$ is a subsemigroup of S.

The proof of this corollary is immediate from Propositions 4.1.4, 4.2.8, and Corollary 4.2.17.

<u>4.2.19: Corollary.</u> Let S be a compact semigroup with unit. The minimal ideal of S is a homomorphic retract of S if and only if the minimal ideal is a group.

Proof: The sufficiency follows immediately from Corollary 4.2.5.

Necessity. Let 1 denote the unit of S. If $e,e' \in K$, then $1e = e = e1$ and $1e' = e' = e'1$. But if K is a homomorphic retract of S, then $K \cap E$ is a reduct of S so that $e = e'$ and hence the cardinality of $K \cap E$ is one. Thus K is a group.

<u>4.2.20: Example.</u> This is an example of a semigroup S containing a homomorphic retract which contains no reduct of S.

Let S be a simple semigroup which contains two distinct minimal left ideals, L and L'. It was shown in Lemma 3.5.2 that $L \cap E$ is a homomorphic retract of S. It

remains only to show that $L \cap E$ contains no reduct of S. Assume the contrary, i.e., $L \cap E$ does contain a reduct G of S. Since $L' \cap E$ is not empty, it contains a point p. Thus, there exists a point g in G such that $pg = gp$. Since $pg \in L$ and $gp \in L'$, then $L' \cap L$ is not empty and so $L = L'$. This is a contradiction. Therefore, G does not exist.

3. The Second Theorem.

Theorem 3.5.15 gives a set of necessary and sufficient conditions for K to be agreeable with respect to S. Since Proposition 3.2.2 says that if K is a homomorphic retract of S, then K is agreeable with respect to S, it must be the case that a subset of the conditions in 3.5.15 are necessary and sufficient for K to be a homomorphic retract of S. Such a theorem is proved in this section.

4.3.1: Lemma. Let g and g' be functions from S into K. The following are equivalent:

(1) $g(x)S = xkS$ and $Skx = Sg'(x)$ for all $x \in S$ and $k \in K$.

(2) $xK = g(x)S$ and $Sg'(x) = Kx$ for all $x \in S$.

Proof: (1) implies (2). Clearly, xK is a subset of $\cup \{xkS | k \in K\} = g(x)S$. Since xK is a right ideal and $g(x)S$ is a minimal right ideal, $xK = g(x)S$. A dual argument demonstrates that $Kx = Sg'(x)$.

(2) implies (1). If $x \in S$ and $k \in K$, then $xk \in g(x)S \cap xkS$ so that $xkS = G(x)S$. A dual argument shows that $Skx = Sg'(x)$.

4.3.2: Theorem. The following are equivalent:

(1) K is a homomorphic retract of S.

(2) There exist functions g and g' from S into K such that $xK = g(x)S$ and $Kx = Sg(x)$ for all $x \in S$.

Proof: (1) implies (2). Since K is a homomorphic retract of S, there exists a reduction f of S onto $K \cap E$. Using the reduction properties of f and Corollary 4.2.16, it is seen that $xkS = xkf(xk)S = f(xk)xkS = f(xk)S = f(f(x)f(k))S = u(f(x)f(k))S = f(x)f(k)S = f(x)S$ for all $x \in S$ and $k \in K$. In a similar fashion, it is seen that $Skx = Sf(x)$ for all $x \in S$ and $k \in K$. Let $g = g' = f$. (2) follows from Lemma 4.3.1.

(2) implies (1). It is enough to prove the function f defined by $f(x) = u(g(x)g'(x))$ is a reduction of S onto $K \cap E$. Clearly, f is a function from S into $K \cap E$, so that it is only necessary to show that f satisfies 4.2.7.1 and 4.2.7.2.

Since $xf(x) \in xf(x)S = g(x)S = g(x)g'(x)S = u(g(x)g'(x))S = f(x)S$, it follows that $xf(x) = f(x)xf(x)$. Using a dual argument, it may be seen that $f(x)x = f(x)xf(x)$ and hence $f(x)x = xf(x)$.

If $e \in K \cap E$, if $x \in S$, and if $xe = ex$, then $ex \in Sex \cap Se = Sg'(x)$ so that $Se = Sg'(x)$. Similarly, $eS = g(x)S$. Hence, $g(x)g'(x) \in eS \cap Se$ and thus $f(x) = u(g(x)g'(x)) = e$.

It remains only to show that $f(xy)xyf(xy) = f(x)xyf(y)$ for all $x,y \in S$. First it is shown that

$f(xy)f(x) = f(x)$ and $f(y)f(xy) = f(y)$. Since $g(xy)S = xyf(x)S = g(x)S$ and $Sg'(xy) = Sf(x)xy = Sg'(x)$, $f(xy) \in g(x)S \cap Sg'(y) = f(x)S \cap Sf(y)$. Thus $f(xy)S = f(x)S$ and $Sf(xy) = Sf(y)$ and hence $f(xy)f(x) = f(x)$ and $f(y)f(xy) = f(y)$ for all $x,y \in S$. Next, since $f(xy)x \in Sf(xy) = Sg'(x) = Sf(x)$, it follows that $f(xy)x = f(xy)xf(x) = f(xy)f(x)x = f(x)x$. In a similar fashion, it follows that $yf(xy) = yf(y)$. Thus $f(xy)xyf(xy) = f(x)xyf(y)$.

4.3.3: Corollary. The following are equivalent:

(1) $K \cap E$ is a reduct of S.

(2) There exists a function f from S to $K \cap E$ such that $xK = f(x)S$ and $Kx = Sf(x)$ for all $x \in S$.

Proof: (1) implies (2). It was shown in the proof of Theorem 4.3.2 that a reduction of S onto $K \cap E$ will satisfy (2).

(2) implies (1). (2) implies K is a homomorphic retract of S which implies $K \cap E$ is a reduct of S.

CHAPTER V

APPLICATION OF HOMOMORPHIC RETRACTIONS TO RELATIVE IDEALS

Let S be a semigroup and let A and T be non-empty subsets of S. Then A is a <u>T-ideal</u> if and only if $TA \cup AT$ is a subset of A, A is <u>right T-ideal</u> if and only if AT is a subset of A, and A is a <u>left T-ideal</u> if and only if TA is a subset of A. A T-ideal is also known as a relative ideal and they have been studied in [17], [18], and [19].

The main problem studied in this chapter is to find conditions which will insure that a minimal T-ideal is a retract of S. The standing hypothesis in Sections 3 and 4 is that T is a homomorphic retract of S. In Section 3 there are propositions which say that if T is a homomorphic retract of S and if one of five other conditions are satisfied, then a minimal T-ideal is a retract of S. In section 4 it is shown that if L and R are minimal left and right T-ideals, respectively, and if both L and R are subsemigroups, then the minimal T-ideal LR is a retract of S in case T is a homomorphic retract of S.

In Section 1 various examples of minimal T-ideals are presented and Section 2 contains some technical propositions which are used in Sections 3 and 4.

1: Examples of Minimal T-ideals.

5.1.1: Example. Let $S = [-1,1]$ under the usual multiplication and let $T = \{-1,1\}$. Then $\{-1,1\}$, $\{0\}$, and $\{-\frac{1}{2},\frac{1}{2}\}$ are minimal T-ideals and the second is a retract of S.

5.1.2: Example. Let $X = [0,1]$ under the multiplication $xy = x$, let $P = \left\{\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} : 0 \le x \le 1\right\} \cup \left\{\begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix} : 0 \le y \le 1\right\}$, let $S = X \times P$, and let $Q = \left\{\left[1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right], \left[0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right]\right\}$. Then $\left\{\left[1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right] \left[0, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\right]\right\}$ is a minimal Q-ideal. If $T = X \times \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\}$, then $X \times \left\{\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}\right\}$ is a minimal T-ideal which is a retract of S.

5.1.3: Example. Let X and P be as in t.1.2, let $Y = Z = [0,1]$ with the multiplication in Y given by $xy = y$, and let Z have the usual multiplication. Let $S = P \times X \times G \times Y \times Z$ where G is any compact topological group. If $T = \left\{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right\} \times X \times G \times Y \times \{1\}$, then any minimal T-ideal is a retract of S.

5.1.4: Example. Let X and Y be as in Example 5.1.3, let $Z = [0,1]$ with the multiplication $xy = \min(x,y)$, and let $S = X \times Z \times Y$ with co-ordinatewise multiplication. If $T = X \times \{1\} \times Y$, then the collection of minimal T-ideals is $\{X \times \{z\} \times Y | z \in Z\}$; each element of this collection is a retract of S. The collections of minimal left and right T-ideals are $\{X \times \{z\} \times \{y\} | z \in Z,\ y \in Y\}$ and $\{\{x\} \times \{z\} \times Y | x \in X,\ z \in Z\}$, respectively; each element of these two collections is a subsemigroup of S.

5.1.5: Example. Let $S = E^2$ with the usual complex number

multiplication. If $T = S^1$, then T is a minimal T-ideal; if $Q = \{-1,1\}$, then $\{-\frac{1}{2},\frac{1}{2}\}$ is a minimal Q-ideal but neither T nor $\{-\frac{1}{2},\frac{1}{2}\}$ is a retract of S.

<u>5.1.6: Example</u>. Let $X = S^1$ with multiplication $xy = x$, let $Z = [0,1]$ with the usual multiplication, and let $S = (X \times Z)/(X \times \{0\})$. If $T = \{(x,1) | 0 \leq \arg x \leq \frac{\pi}{2}\}$, then $\{(x,\frac{1}{2}) | 0 \leq \arg x \leq \frac{\pi}{2}\}$ is a minimal T-ideal and an absolute retract.

<u>5.1.7: Example.</u> This example shows that there exists a semigroup S which contains a homomorphic retract T and a minimal T-ideal I which is a retract of S, but I is not a homomorphic retract of S. Moreover, $I = LR$ where L and R are minimal left and right T-ideals which are intersecting subsemigroups of S.

Let $P = \left\{\begin{bmatrix} x & 0 \\ 0 & 1 \end{bmatrix} \middle| x \text{ is a real number and } 0 \leq x \leq 1\right\} \cup \left\{\begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix} \middle| y \text{ is a real number and } 0 \leq y \leq 1\right\}$, let X be $[0,1]$ with the usual multiplication, let Y be $[0,1]$ with the multiplication given by $xy = \min(x,y)$, and let $S = X \times P \times Y$ with co-ordinatewise multiplication. If $T = X \times P \times \{1\}$, then T is a homomorphic retract of S and $I = X \times K \times \{\frac{1}{2}\}$ is a minimal T-ideal where $K = \left\{\begin{bmatrix} 0 & y \\ 0 & 1 \end{bmatrix} \middle| y \text{ is a real number and } 0 \leq y \leq 1\right\}$.

To see that I is not a homomorphic retract of S, we assume the contrary, i.e., there exists a homomorphic retraction f of S onto I. Using the homomorphic retraction properties of f, it is seen that $\left(1, \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right) = \left(1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, 1\right)$

$\left(1, \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right) = f\left(1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, 1\right) f\left(1, \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right) =$ $f\left(1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, 1\right) \left(1, \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right)$. Since $f\left(1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, 1\right) \in I$, it is of the form $(x,k,\frac{1}{2})$ where $x \in X$ and $k \in K$. Therefore $(x,k,\frac{1}{2}) \left(1, \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right) = (x,k,\frac{1}{2})$; thus, $f\left(1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, 1\right) =$ $\left(1, \begin{bmatrix} 0 & \frac{1}{4} \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right)$. Similarly, it is seen that $\left(1, \begin{bmatrix} 0 & \frac{1}{2} \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right) =$ $\left(1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, 1\right) \left(1, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right) = f\left(1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, 1\right) \left(1, \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, \frac{1}{2}\right)$ $= f\left(1, \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}, 1\right)$, which is a contradiction.

2: Preliminary Propositions.

Throughout the rest of this chapter, S is a compact semigroup, T is a non-empty subset of S, and f is a homomorphic retraction of S onto T. It should be noted that this implies T is a closed subsemigroup of S. The minimal ideal of T is denoted by K, H is the union of all subgroups of S, E is the set of all idempotents of S, u is the function from H onto E defined by "$u(x)$ is the unit of a group which contains x" [20], and v is the function from H onto H defined by "$v(x)$ is the inverse of x in a group which contains x" [20].

5.2.1: Lemma. If I is a minimal T-ideal, then $f(I) = K$.

Proof: It follows from the Homomorphism Theorem [15] that $f(I)$ is an ideal of T and so K is a subset of $f(I)$. Since $K \cap E$ is not empty, there exists a point a in I such that $f(a) \in E \cap K$. The fact that $Tf(a)af(a)T$ is a T-ideal contained in I implies that $Tf(a)af(a)T = I$. Thus, using the

homomorphic retraction properties of f, it is seen that $f(1) = f(Tf(a)af(a)T) = f(T)f^2(a)f(a)f^2(a)f(T) = Tf(a)f(a)f(a)T = Tf(a)T = K$.

<u>5.2.2: Proposition.</u> If I is a minimal T-ideal, then $K \cap E$ is a reduct of $K \cup (\cup\{I^n | n = 1,2,\ldots\})$. Moreover, if r is the reduction of $K \cup (\cup\{I^n | n = 1,2,\ldots\})$ onto $K \cap E$, then $r(x)x = xr(x) = x$ for all $x \in K \cup (\cup\{I^n | n = 1,2,\ldots\})$.

Proof: Let $r = uf$; it must be shown that r has the following properties.

(1) $r(x)x = xr(x) = x$ for all $x \in K \cup (\cup\{I^n | n = 1,2,\ldots\}$.

(2) If $e \in K \cap E$ and $ex = xe$, then $r(x) = e$.

(3) If $x,y \in K \cup (\cup\{I^n | n = 1,2,\ldots\})$, then $r(xy)$ $xyr(xy) = r(x)xyr(y)$.

It should be noted that (3) follows immediately from (1).

If $x \in K$, then (1) and (2) are clear. Suppose $x \in \cup\{I^n | n = 1,2,\ldots\}$. Then $x = x_1 x_2,\ldots x_n$, where $x_i \in I$ for $i = 1,2,3,\ldots n$. From Lemma 5.2.1, it is seen that there exists a minimal left ideal L_o of T and a minimal right ideal R_o of T such that $f(x) \in R_o \cap L_o$. $I = L_o x_1 R_o$, there exist t_1, $t_n \in L_o$ and p_1, $p_n \in R_o$ such that $t_1x_1p_1 = x_1$ and $t_nx_1p_n = x_n$. Thus $f(x) = t_1\ f(x_1p_1x_2\ldots t_nx_1)p_n \in (L_o \cap R_o) \cap (t_1S \cap Sp_n)$ so that $t_1S = R_o$ and $Sp_n = L_o$. It follows that $t_1, P_n \in L_o \cap R_o$ and so $[uf(x)]x = uf(x)t_1x_1p_1x_2\ldots x_{n-1}t_nx_1p_n = t_1x_1p_1\ldots x_{n-1}x_1p_n = x = x[uf(x)]$.

<u>5.2.3: Corollary.</u> If $a \in K \cup (\cup\{I^n | n = 1,2,3,\ldots\}$ and $f(a) \in E$, then $f(a) \in f(a)Sf(a)$ and $f(a)a = af(a) = a$.

Proof: If r is defined as in Proposition 5.2.2, then $r(a) = uf(a) = f(a)$ and the corollary follows from Proposition 5.2.2.

<u>5.2.4: Lemma.</u> If L is a minimal left T-ideal, then $f(L)$ is a minimal left ideal of T.

Proof: Since $f(L)$ is a left ideal of T, it contains a minimal left ideal L_o of T. Because L_o is not empty, there exists a point a in L such that $f(a) \epsilon L_o$. Thus $L_o = Tf(a) = f(T)f(a) = f(Ta) = f(L)$.

<u>5.2.5: Lemma.</u> If R is a minimal right T-ideal, then $f(R)$ is a minimal right ideal of T.

The proof is dual to that of 5.2.4.

<u>5.2.6: Lemma.</u> Let L_o and R_o be minimal left and minimal right ideals of T, respectively. If $t_1 \epsilon L_o$, if $t_2 \epsilon R_o$, if $u(L_o \cap R_o) = e$, and if $t_1 t_2 = e$, then $t_1 = v(t_2)$.

Proof: Since $t_1 t_2 = e \epsilon L_o \cap R_o \cap t_1 T \cap Tt_2$, then $t_1 \epsilon t_1 T = R_o$ and $t_2 \epsilon Tt_2 = L_o$. Thus t_1 and t_2 are in $L_o \cap R_o$ and $t_1 = v(t_2)$.

<u>5.2.7: Lemma.</u> Let I be a minimal T-ideal and let $a \epsilon I$ such that $f(a) \epsilon E$. If $x \epsilon I$ and $f(x) = f(a)$, then there exists a point $t \epsilon K$ such that $x = txv(t)$. Moreover, if $x = t_1 a t_2$ where $t_1 \epsilon Tf(a)$ and $t_2 \epsilon f(a)T$, then $t_1 = v(t_2)$.

Proof: Since $I = Tf(a)af(a)T$, there exist points $t_1 \epsilon Tf(a)$ and $t_2 \epsilon f(a)T$ such that $x = t_1 a t_2$. Thus $f(a) = f(x) = f(t_1 a t_2) = t_1 f(a) t_2 = t_1 t_2$ and so it follows from Lemma 5.2.5 that $t_1 = v(t_2)$.

5.2.8: Proposition. If L is a minimal left T-ideal, then f restricted to L is a one-to-one function.

Proof: Since $f(L)$ is a minimal left ideal of T, there exists $a \in L$ such that $f(a)$ is idempotent. If $x,y \in L$, then $f(L)a = L$ implies that there exist $t_1, t_2 \in f(L)$ such that $x = t_1 a$ and $y = t_2 a$. If $f(x) = f(y)$, then $t_1 = t_1 f(a) = f(t_1 a) = f(x) = f(y) = f(t_2 a) = t_2 f(a) = t_2$ because f is a homomorphic retraction.

5.2.9: Proposition. If R is a minimal right T-ideal, then f restricted to R is a one-to-one function.

The proof is dual to the proof of Proposition 5.2.7.

5.2.10: Corollary. If L is a minimal left T-ideal, if R is a minimal right T-ideal, and if $L \cap R$ is not empty, then f restricted to R is a one-to-one function.

The proof is clear.

5.2.11: Proposition. If L and L' are minimal left T-ideals and if $f(L) \cap f(L')$ is not empty, then $f(L) = f(L')$.

Proof: This follows immediately from the fact that $f(L)$ and $f(L')$ are minimal left ideals of T.

The following is the dual statement of the previous proposition.

5.2.12: Proposition. If R and R' are minimal right T-ideals and if $f(R) \cap f(R')$ is not empty, then $f(R) = f(R')$.

3: Various Sufficient Conditions.

In this section, it is shown that if T is a homomorphic retract of S, if I is a minimal T-ideal of S, and

if one of the following five conditions holds, then I is a retract of S:

(1) I has a cut point,

(2) f restricted to I is one-to-one,

(3) K is contained in E,

(4) T is normal in S, and

(5) $H = \{(x,y) | x \cup Tx = y \cup Ty \text{ and } y \cup xT = y \cup yT\}$ is a congruence of S.

<u>5.3.1: Proposition.</u> If I is a minimal T-ideal and if f restricted to I is one-to-one, then I is a retract of S.

Proof: It has been shown that there exists a retraction q of T onto K [20]. The function $(f|I)^{-1}qf$ is a retraction of S onto I since if $x \in I$, then $(f|I)^{-1}qf(x) = (f|I)^{-1}f(x) = x$.

<u>5.3.2: Corollary.</u> If I is a minimal T-ideal and if K is a subset of E, then I is a retract of S.

Proof: It is sufficient to show that f restricted to I is a one-to-one function.

Let $a \in I$, let $L = Tf(a)$, and let $R = f(a)T$, then $K = LR$ and $I = LaR$. If $x,y \in I$ and if $f(x) = f(y)$, then there exist t_1, $t_1' \in L$ and t_2, $t_2' \in R$ such that $x = t_1at_2$ and $y = t_1'at_2'$. The fact that f(a) is a right unit for L implies that $t_1t_2 = f(t_1at_2) = f(t_1'at_2') = t_1't_2'$ so that $t_1T = t_1'T$ and $Tt_2 = Tt_2'$. Because of the hypothesis that K is a subset of E, $\{t_1\} = t_1T \cap Tt_1 = t_1'T \cap Ta = t_1'T \cap Tt_1' = \{t_1'\}$. In a similar fashion, it is seen that $t_2 = t_2'$ and hence $x = t_1at_2 = t_1'at_2' = y$; thus f restricted to I is one-to-one.

<u>5.3.3: Theorem.</u> If I is a minimal T-ideal and if there is a point a in I such that Ta is a subset of aT, then I is a minimal left T-ideal.

Proof: Since I is the union of minimal left T-ideals [15], there exists a minimal left T-ideal L such that $a \in L$. Thus $L = Ta$ and $I = TaT \subset TTa \subset Ta = L$. But L is a subset of I so that $L = I$.

<u>5.3.4: Theorem.</u> If I is a minimal T-ideal and if there exists $a \in I$ such that aT is a subset of Ta, then I is minimal right T-ideal.

The proof is dual to that of Theorem 5.3.3.

<u>5.3.5: Corollary.</u> If I is a minimal T-ideal and if T is normal in I, i.e., $xT = Tx$ for all $x \in I$, then I is a minimal left and a minimal right T-ideal.

It should be noted that the only assumption about T in 5.3.3, 5.3.4, and 5.3.5 is that T is a closed subsemigroup of S.

<u>5.3.6: Corollary.</u> If I satisfies the hypotheses of 5.3.3 or of 5.3.4, then I is a retract of S.

Proof: By 5.3.3, I is a minimal left T-ideal and hence by 5.2.8, f restricted to I is one-to-one. Thus, 5.3.1 implies that I is a retract of S.

<u>5.3.7: Corollary.</u> If I is a minimal T-ideal and if there exists a point a in I such that $Ta = a$, then I is a retract of S.

Proof: Since I is the union of minimal left T-ideals and the union of minimal right T-ideals [17], there exist a

minimal left T-ideal L and a minimal right T-ideal R such that $a \in R \cap L$. Therefore, $L = Ta$ and $R = aT$. Thus $Ta = \{a\} \subset aT$ and the hypotheses of 5.3.3 are satisfied so that it follows from Corollary 5.3.6 that I is a retract of S.

<u>5.3.8: Corollary.</u> If I is a minimal T-ideal and if there exists a point a in I such that $aT = a$, then I is a retract of S.

The proof is dual to that of Corollary 5.3.7.

<u>5.3.9: Corollary.</u> If I is a minimal T-ideal and if I has a cut point, then I is a retract of S.

Proof: It follows from Faucett's Theorem [15] that $aT = A$ or $Ta = a$ for all a in I. Thus the hypotheses of 5.3.7 or of 5.3.8 are satisfied and the conclusion follows:

Recall that $L(x) = x \cup Tx$, $R(x) = x \cup xT$, $L_x = \{y|L(x) = L(y)\}$, $R_x = \{y|R(x) = R(y)$, $H_x = L_x \cap R_x$, $\mathcal{H} = \{(x,y)|x \in H_y\}$, and H is a closed equivalence relation on S [18]. In the rest of this section it is assumed that $\mathcal{H}$ is a congruence of S.

Notation: Let $S^* = S/\mathcal{H}$, let φ be the projection of S onto S^*, let $T^* = \varphi(T)$, let $I^* = \varphi(I)$, and let $K^* = \varphi(K)$.

It follows from Proposition 4.1.5 that there exists a homomorphic retraction f^* of S^* onto T^* such that the following diagram commutes.

$$\begin{tikzcd} S^* \arrow[r, "f^*"] & S^* \\ S \arrow[u, "\varphi"] \arrow[r, "f"] & S \arrow[u, "\varphi"'] \end{tikzcd}$$

<u>5.3.10: Lemma.</u> If I is a minimal I-ideal, then I* is a minimal T*-ideal.

Proof: Let J* be a minimal T*-ideal contained in I* and let $a \in I$ such that $\varphi(a) \in J^*$. Then $J^* = T^*\varphi(a)T^* = \varphi(T)\varphi(a)\varphi(T) = \varphi(TaT) = \varphi(I) = I^*$.

It should be noted that K* is the minimal ideal of T* and every point in K* is an idempotent.

<u>5.3.11: Theorem.</u> If $\mathcal{H}$ is a congruence of S, the I is a retract of S.

Proof: It is sufficient to show that f restricted to I is one-to-one.

If $x, y \in I$ and $f(x) = f(y)$, then $f^*\varphi(x) = f^*\varphi(y)$. But $\varphi(x) = \varphi(y)$ by Corollary 5.3.4. Thus $y \in H_x \subset R_x = Tx$ since x is an element of a minimal right T-ideal. Therefore, f restricted to Tx one-to-one implies that $x = y$.

<u>4: The Subsemigroup Case.</u>

In this section, L and R are minimal left and right T-ideals, respectively, both of which are subsemigroups of S and such that $R \cap L$ is not empty. In [15] it is proved that $I = LR$ is a minimal T-ideal. In this section, it is shown that I is a retract of S.

Since $R \cap L$ is not empty, it is a group and has a unique idempotent a. If $f(R) = R_o$ and $f(L) = L_o$, then $L = L_o a$, $R = aR_o$, and $I = L_o a R_o = LaR = LR$.

In the following two propositions the hypothesis that T is a homomorphic retract of S may be replaced with the hypothesis that T is a closed subsemigroup of S.

<u>5.4.1: Proposition.</u> If L' is a minimal left T-ideal contained in I, then L' is a subsemigroup of S.

Proof: If $x \in L' \subset I = L_oaR_o$, then there exist $t_1 \in L_o$ and $t_2 \in R_o$ such that $x = t_1at_2$. Since $L' = L_ox = L_ot_1at_2 = L_oat_2 = Lt_2$ and the left T-ideal L is a subsemigroup, it follows that $L'L' = Lt_2Lt_2 \subset LLt_2 \subset Lt_2 = L'$.

<u>5.4.2: Proposition.</u> If R' is a minimal right T-ideal contained in I, then R' is a subsemigroup of S.

The proof of this proposition is dual to that of Proposition 5.4.1.

<u>5.4.3: Lemma.</u> If $x \in I$ and if $f(x) = f(a)$, then $x = a$.

Proof: Since $I = L_oaR_o$, there exist $t_1 \in L_o$ and $t_2 \in R_o$ such that $x = t_1at_2$. It follows from Lemma 5.2.6 that $t_1 = v(t_2)$.

Using [15, Proposition 20B] it is seen that $t_1a = t_1af(a) \in R_oaL_o = (R_oa)(aL_o) = RL = R \cap L$. Since $R \cap L$ is a group, there exists a unique point $k \in R \cap L$ such that $t_1ak = a$; $k = at$ for some $t \in R_o$ since $k \in R = aR_o$. Because f is a homomorphic retraction, $f(k) = f(at) = f(a)t = t$ so that $k = af(k)$. Thus $f(a) = f(t_1ak) = t_1f(k)$ and hence $f(k) = v(t_1) = t_2$. It follows that $x = t_1at_2 = t_1aat_2 = t_1aaf(k) = t_1ak = a$.

<u>5.4.4: Proposition.</u> If L' and L'' are minimal left T-ideals and if $f(L'') \cap f(L')$ is not empty, then $L'' = L'$.

Proof: If $x \in L'$, then there exists a minimal right T-ideal R' contained in I such that $x \in L' \cap R'$ [15, Proposition 20A]. From Lemmas 5.4.1, 5.4.2, and

[15, Proposition 20 B], it follows that $R' \cap L'$ is group whose unique idempotent is denoted by e.

From Proposition 5.2.10, it follows that $f(L'') = f(L')$ and so there exists a point $y \in L''$ such that $f(y) = f(e)$. But by Lemma 5.4.3, $y = e$, which means $L'' \cap L'$ is not empty. Therefore, $L'' = L'$ since both L'' and L' are minimal left T-ideals.

5.4.5: Proposition. If R' and R'' are minimal right T-ideals contained in I, and if $f(R') \cap f(R'')$ is not empty, then $R' = R''$.

The proof of this proposition is dual to the proof of Proposition 5.4.4.

5.4.6: Theorem. The minimal T-ideal I is a retract of S.

Proof: If $x,y \in I$ and $f(x) = f(y)$, then there exist minimal left T-ideals L' and L'' such that $x \in L'$ and $y \in L''$. Thus $f(L') \cap f(L'')$ is not empty so that it follows from 5.4.4 that $L' = L''$. Since f restricted to L' is one-to-one, $x = y$ and hence f restricted to I is one-to-one. Hence the conclusion follows from Theorem 5.3.3.

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BIOGRAPHICAL SKETCH

Joseph Thomas Borrego, Jr. was born September 30, 1939 in Tampa, Florida. He was educated in the public school system of Tampa and graduated from Thomas Jefferson High School in June, 1957.

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This dissertation was prepared under the direction of the Chairman of the Candidate's supervisory committee and has been approved by all members of that committee. It was submitted to the Dean of the College of Arts and Sciences and to the Graduate Council, and was approved as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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